Information flow within stochastic dynamical systems

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Information flow or information transfer is an important concept in general physics and dynamical systems which has applications in a wide variety of scientific disciplines. In this study, we show that a rigorous formalism can be established in the context of a generic stochastic dynamical system. An explicit formula has been obtained for the resulting transfer measure, which possesses a property of transfer asymmetry and, if the stochastic perturbation to the receiving component does not rely on the giving component, has a form the same as that for the corresponding deterministic system. This formula is further illustrated and validated with a two-dimensional Langevin equation. A remarkable observation is that, for two highly correlated time series, there could be no information transfer from one certain series, say x_2, to the other (x_1). That is to say, the evolution of x_1 may have nothing to do with x_2, even though x_1 and x_2 are highly correlated. Information flow analysis thus extends the traditional notion of correlation analysis and/or mutual information analysis by providing a quantitative measure of causality between dynamical events.

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Information flow is a fundamental notion in general physics, and quantification of this notion has been a continuing problem in the physics community [1–5]. Practical applications have been reported in fields like neuroscience [6] and atmosphere-ocean science [7], and are envisioned in the diverse disciplines such as turbulence research, material science, and nanotechnology, to name a few, where ensemble forecasts [8] are involved and predictability [9] becomes an issue. Recently, Liang and Kleeman [3,4] established for this important concept a rigorous formalism in the context of deterministic dynamical systems. In this study, we will show such a formalism can also be obtained if stochasticity is taken into account. We consider two-dimensional (2D) systems only; systems of higher dimensionality will be reported elsewhere [10].

We start in this paragraph with a brief review of the work in [3] to educe the strategy for the building of our formalism for stochastic systems. Consider a 2D system

$$\frac{d\underline{\mathbf{x}}}{dt} = \underline{\mathbf{F}}(\underline{\mathbf{x}},t), \tag{1}$$

where $\underline{\mathbf{F}}=(F_1,F_2)$, and the state variables $\underline{\mathbf{x}}=(x_1,x_2)\in\mathbb{R}^2$. The randomness is limited within the initial condition. For simplicity, we follow the convention of notation in the physics literature and do not distinguish random variables and deterministic variables, which should be clear in the context. (In probability theory, they are usually distinguished with lower and upper cases.) Let ρ be the joint probability density of x_1 and x_2, and suppose that it and its derivatives have compact support. Without loss of generality, consider the information transfer from x_2 to x_1. We need the marginal density of x_1, $\rho_1(t;x_1)=\int_{\mathbb{R}}\rho dx_2$, and the marginal (Shannon) entropy, $H_1=-\int_{\mathbb{R}}\rho_1 \log \rho_1 dx_1$. H_1 varies as the system moves forward. Its variation is due to two different mechanisms, one due to x_1 itself, written as dH_1^*/dt, another due to the transfer from x_2. The latter is the very information transfer, which we will write as $T_{2\to1}$ hereafter. The rate of information transfer from x_2 to x_1 is therefore the difference between dH_1/dt and dH_1^*/dt, $T_{2\to1}=dH_1/dt-dH_1^*/dt$. Among the terms on the right-hand side, dH_1/dt can be derived from the Liouville equation [12] corresponding to Eq. (1); the key is the derivation of dH_1^*/dt, the entropy change as x_1 evolves on its own. In [3], this is achieved with the aid of a theorem established therein: The joint entropy of (x_1,x_2), $H=-\iint_{\mathbb{R}^2}\rho \log \rho d\underline{\mathbf{x}}$, evolves as

$$\frac{dH}{dt} = E(\nabla\cdot\underline{\mathbf{F}}). \tag{2}$$

Here the operator E is the mathematical expectation with respect to ρ. [Equation (2) holds not just for 2D systems; it is actually true for systems of arbitrary dimensionality.] Liang and Kleeman then intuitively argued that

$$\frac{dH_1^*}{dt} = E\left(\frac{\partial F_1}{\partial x_1}\right), \tag{3}$$

a result later on they rigorously proved [4], and hence obtained the transfer $T_{2\to1}$.

The above formalism has been generalized to the information transfer within a deterministic system of arbitrary dimensionality [4]; the key equation (3) has also been used to form the transfer between two subspaces [5]. The generalization, however, encounters difficulty if stochasticity is involved. Consider a system

$$d\underline{\mathbf{x}} = \underline{\mathbf{F}}(\underline{\mathbf{x}},t)dt + \underline{\mathbf{B}}(\underline{\mathbf{x}},t)d\underline{\mathbf{w}}, \tag{4}$$

where $\underline{\mathbf{w}}=(w_1,w_2)$ is a standard 2D Wiener process ($d\underline{\mathbf{w}}/dt$ is usually referred to as "white noise"), and $\underline{\mathbf{B}}=(b_{ij})$ is the perturbation amplitude. There is no such elegant form as Eq. (2) for the evolution of H. One thus cannot obtain dH_1^*/dt intuitively as Eq. (3) is obtained.

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On the other hand, dH_1^*/dt may be equally understood as the rate of change of the marginal entropy of x_1 with the effect from x_2 excluded. This alternative interpretation, as we used in [4], sheds light on the above problem. To reflect this interpretation, we will denote the term as $dH_{1\not{2}}/dt$ henceforth, the subscript $\not{2}$ signifying "x_2 excluded." The rate of information transfer from x_2 to x_1 is thence

$$T_{2\to 1} = \frac{dH_1}{dt} - \frac{dH_{1\not{2}}}{dt}. \tag{5}$$

Here the key issue is how to find $dH_{1\not{2}}/dt$, which we will show shortly after the evaluation of dH_1/dt.

To find dH_1/dt, we need to know density evolution. Corresponding to Eq. (4) there is a Fokker-Planck equation [12]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (F_1\rho)}{\partial x_1} + \frac{\partial (F_2\rho)}{\partial x_2} = \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial^2 (g_{ij}\rho)}{\partial x_i \partial x_j}, \tag{6}$$

where $g_{ij}=g_{ji}=\Sigma_{k=1}^{2} b_{ik}b_{jk}$, $i,j=1,2$. This integrated over $\mathbb{R}$ with respect to x_2 gives the evolution of ρ_1:

$$\frac{\partial \rho_1}{\partial t} + \int_{\mathbb{R}} \frac{\partial (F_1\rho)}{\partial x_1} dx_2 = \frac{1}{2}\int_{\mathbb{R}} \frac{\partial^2 (g_{11}\rho)}{\partial x_1^2} dx_2. \tag{7}$$

Note in the derivation we have used the fact that ρ and its derivatives vanish at the boundaries as they are compactly supported. For notational succinctness, we will henceforth suppress the integral domain $\mathbb{R}$, unless otherwise noted. Multiplying Eq. (7) by $-(1+\log \rho_1)$ followed by an integration with respect to x_1 over $\mathbb{R}$, one obtains

$$\frac{dH_1}{dt} - \int\int \log \rho_1 \frac{\partial (F_1\rho)}{\partial x_1} dx_1 dx_2 = -\frac{1}{2}\int\int \log \rho_1 \frac{\partial^2 (g_{11}\rho)}{\partial x_1^2} dx_1 dx_2.$$

Integrating by parts, this is reduced to

$$\frac{dH_1}{dt} = -E\left(F_1 \frac{\partial \log \rho_1}{\partial x_1}\right) - \frac{1}{2}E\left(g_{11}\frac{\partial^2 \log \rho_1}{\partial x_1^2}\right), \tag{8}$$

where E stands for expectation with respect to ρ.

The key part of this study is the evaluation of $H_{1\not{2}}$. Examine a small time interval $[t,t+\Delta t]$. $H_{1\not{2}}(t+\Delta t)$ is the marginal entropy of x_1 at time $t+\Delta t$ as x_2 is frozen as a parameter instantaneously at t. One thence needs to consider a system on $[t,t+\Delta t]$ suddenly modified at time t from that prior to t. Clearly, $H_{1\not{2}}$ cannot be derived from the Fokker-Planck equation (7), where the dynamics is consistent through time. One has to go back to the definition of derivative to achieve the goal. Let the marginal entropy evolved from t to $t+\Delta t$ with x_2 frozen at t be $H_{1\not{2}}(t+\Delta t)$. We then have

$$\frac{dH_{1\not{2}}}{dt} = \lim_{\Delta t\to 0} \frac{H_{1\not{2}}(t+\Delta t) - H_1(t)}{\Delta t},$$

and the whole problem now boils down to the derivation of $H_{1\not{2}}(t+\Delta t)$. In [4], we discretize the deterministic equation (1) and evaluate the Frobenius-Perron operator for the discretized system to compute the modified marginal entropy. For the stochastic system (4), however, there is no such simple operator. We need a different approach for the problem.

Denote by $x_{1\not{2}}$ the first component after x_2 is fixed as a parameter. The stochastic system (4) is changed to

$$dx_{1\not{2}} = F_1(x_{1\not{2}},x_2,t)dt + \sum_k b_{1k}dw_k \quad \text{on } [t,t+\Delta t], \tag{9}$$

$$x_{1\not{2}} = x_1 \quad \text{at time } t. \tag{10}$$

Correspondingly the density $\rho_{1\not{2}}$ evolves following the following Fokker-Planck equation:

$$\frac{\partial \rho_{1\not{2}}}{\partial t} + \frac{\partial (F_1\rho_{1\not{2}})}{\partial x_1} = \frac{1}{2}\frac{\partial^2 (g_{11}\rho_{1\not{2}})}{\partial x_1^2}, \quad t \in [t,t+\Delta t], \tag{11}$$

$$\rho_{1\not{2}} = \rho_1 \quad \text{at } t, \tag{12}$$

where $g_{11}=\Sigma_k b_{1k}^2$. Recall by definition, the Shannon entropy may be understood as the expectation of a function of the state variable formed by minus logarithm composite with its density. This motivates one to introduce a function of x_1, $f_t(x_1)=\log \rho_{1\not{2}}(t,x_1)$, whose evolution is obtained by dividing Eq. (11) by $\rho_{1\not{2}}$:

$$\frac{\partial f_t}{\partial t} + \frac{1}{\rho_{1\not{2}}}\frac{\partial F_1\rho_{1\not{2}}}{\partial x_1} = \frac{1}{\rho_{1\not{2}}}\frac{\partial^2 g_{11}\rho_{1\not{2}}}{\partial x_1^2}.$$

In a discretized version, this is

$$f_{t+\Delta t}(x_1) = f_t(x_1) - \frac{\Delta t}{\rho_1}\frac{\partial (F_1\rho_1)}{\partial x_1} + \frac{\Delta t}{2\rho_1}\frac{\partial^2 (g_{11}\rho_1)}{\partial x_1^2} + O(\Delta t^2),$$

where the fact $\rho_{1\not{2}}=\rho_1$ at time t has been used. [Functions without arguments explicitly written out are supposed to be evaluated at $x_1(t)$.] So

$$f_{t+\Delta t}[x_{1\not{2}}(t+\Delta t)] = f_t[x_{1\not{2}}(t+\Delta t)] - \frac{\Delta t}{\rho_1}\frac{\partial (F_1\rho_1)}{\partial x_1} + \frac{\Delta t}{2\rho_1}\frac{\partial^2 (g_{11}\rho_1)}{\partial x_1^2} + O(\Delta t^2).$$

The $x_{1\not{2}}(t+\Delta t)$ in the argument can be expanded by the Euler-Bernstein approximation [12] of Eq. (9):

$$x_{1\not{2}}(t+\Delta t) = x_1(t) + F_1\Delta t + \sum_k b_{1k}\Delta w_k + \text{h.o.t.}$$

("h.o.t." stands for "higher-order terms"). Substituting back and performing Taylor series expansion, we get

$$f_{t+\Delta t}[x_{1\not{2}}(t+\Delta t)] = f_t\left(x_1 + F_1\Delta t + \sum_k b_{1k}\Delta w_k\right) - \frac{\Delta t}{\rho_1}\frac{\partial (F_1\rho_1)}{\partial x_1} + \frac{\Delta t}{2\rho_1}\frac{\partial^2 (g_{11}\rho_1)}{\partial x_1^2} + O(\Delta t^2)$$
$$= f_t(x_1) + \frac{\partial f_t}{\partial x_1}\left(F_1\Delta t + \sum_k b_{1k}\Delta w_k\right)$$

$$+\frac{1}{2}\frac{\partial^2 f_t}{\partial x_1^2}\left(F_1\Delta t+\sum_k b_{1k}\Delta w_k\right)^2 - \frac{\Delta t}{\rho_1}\frac{\partial(F_1\rho_1)}{\partial x_1}+\frac{\Delta t}{2\rho_1}\frac{\partial^2(g_{11}\rho_1)}{\partial x_1^2}+O(\Delta t^2). \tag{13}$$

Take expectation on each side with respect to their respective random variables and equate the expectations (the equality is a fundamental fact that defines the density evolution [12]), the left-hand side is $-H_{1\tilde{2}}(t+\Delta t)$, and the first term on the right-hand side is $-H_1(t)$. Note that $\Delta w_k \sim N(0,\Delta t)$ for a Wiener process w_k. So

$$E\Delta w_k = 0, \quad E(\Delta w_k)^2 = \Delta t.$$

The second term on the right-hand side is

$$\Delta t E\left(F_1\frac{\partial f_t}{\partial x_1}\right)+E\left(\frac{\partial f_t}{\partial x_1}\sum_k b_{1k}\Delta w_k\right)=\Delta t E\left(F_1\frac{\partial f_t}{\partial x_1}\right),$$

where we have used the fact that Δw_k is independent of (x_1,x_2), and hence expectation can be taken inside directly with Δw_k, which eliminates $E(\frac{\partial f_t}{\partial x_1}\Sigma_k b_{1k}\Delta w_k)$. For the same reason, the third term after expansion leaves only one subterm of order Δt, namely,

$$\frac{1}{2}E\left[\frac{\partial^2 f_t}{\partial x_1^2}\sum_k b_{1k}\Delta w_k\sum_j b_{1j}\Delta w_j\right] = \frac{1}{2}E\left[\frac{\partial^2 f_t}{\partial x_1^2}\left(\sum_k b_{1k}^2(\Delta w_k)^2+\sum_{k\neq j} b_{1k}b_{1j}\Delta w_k\Delta w_j\right)\right].$$

Recall that the perturbations are independent. The summation over $k\neq j$ inside the parentheses thus vanishes after expectation is performed. The first summation is equal to $g_{11}\Delta t$, by the definition of g_{ij} and the fact $E(\Delta w_k)^2=\Delta t$. So the whole term is $\frac{\Delta t}{2}E[g_{11}\frac{\partial^2 f_t}{\partial x_1^2}]$. With all these put together, expectation of Eq. (13) gives [note $f_t=\log\rho_{1\tilde{2}}(t;x_1)=\log\rho_1$]

$$H_{1\tilde{2}}(t+\Delta t)=H_1(t)-\Delta t E\left(F_1\frac{\partial\log\rho_1}{\partial x_1}\right)-\frac{\Delta t}{2}E\left(g_{11}\frac{\partial^2\log\rho_1}{\partial x_1^2}\right) + \Delta t E\left(\frac{1}{\rho_1}\frac{\partial(F_1\rho_1)}{\partial x_1}\right)-\frac{\Delta t}{2}E\left(\frac{1}{\rho_1}\frac{\partial^2(g_{11}\rho_1)}{\partial x_1^2}\right) + O(\Delta t^2).$$

The second and fourth terms on the right-hand side can be combined to give $\Delta t E(\frac{\partial F_1}{\partial x_1})$. So

$$\frac{dH_{1\tilde{2}}}{dt}=\lim_{\Delta t\to 0}\frac{H_{1\tilde{2}}(t+\Delta t)-H_1(t)}{\Delta t} = E\left(\frac{\partial F_1}{\partial x_1}\right)-\frac{1}{2}E\left(g_{11}\frac{\partial^2\log\rho_1}{\partial x_1^2}\right) - \frac{1}{2}E\left(\frac{1}{\rho_1}\frac{\partial^2(g_{11}\rho_1)}{\partial x_1^2}\right). \tag{14}$$

In the equation, the second and the third terms on the right-hand side are from the stochastic perturbation. The first term is precisely Eq. (3), the key result obtained in [3] through intuitive argument based on the theorem (2). The above derivation supplies a proof of this argument.

The information transfer from x_2 to x_1 is obtained by subtracting Eq. (14) from Eq. (8):

$$T_{2\to 1}=-E\left(F_1\frac{\partial\log\rho_1}{\partial x_1}\right)-E\left(\frac{\partial F_1}{\partial x_1}\right)+\frac{1}{2}E\left(\frac{1}{\rho_1}\frac{\partial^2(g_{11}\rho_1)}{\partial x_1^2}\right) = -E\left(\frac{1}{\rho_1}\frac{\partial(F_1\rho_1)}{\partial x_1}\right)+\frac{1}{2}E\left(\frac{1}{\rho_1}\frac{\partial^2(g_{11}\rho_1)}{\partial x_1^2}\right), \tag{15}$$

where E is the expectation with respect to $\rho(x_1,x_2)$. Notice that the conditional density of x_2 on x_1, $\rho_{2|1}$, is ρ/ρ_1. If we define an operator $\mathcal{E}_{2|1}$ such that $\mathcal{E}_{2|1}f$ is the expectation of $f=f(x_1,x_2)$ with respect to $\rho_{2|1}$, followed by an integration with x_1 over $\mathbb{R}$, i.e.,

$$\mathcal{E}_{2|1}f=\int\int\rho_{2|1}(x_2|x_1)f(x_1,x_2)dx_1dx_2,$$

then the above formula may be further simplified:

$$T_{2\to 1}=-\mathcal{E}_{2|1}\left(\frac{\partial(F_1\rho_1)}{\partial x_1}\right)+\frac{1}{2}\mathcal{E}_{2|1}\left(\frac{\partial^2(g_{11}\rho_1)}{\partial x_1^2}\right). \tag{16}$$

This is the transfer from x_2 to x_1. Likewise, the transfer from x_1 to x_2 can be obtained:

$$T_{1\to 2}=-\mathcal{E}_{1|2}\left(\frac{\partial(F_2\rho_2)}{\partial x_2}\right)+\frac{1}{2}\mathcal{E}_{1|2}\left(\frac{\partial^2(g_{22}\rho_2)}{\partial x_2^2}\right), \tag{17}$$

where $\rho_2=\int\rho dx_1$ is the marginal density of x_2.

Among the two terms of Eq. (16) the first is the same in form as the information transfer obtained in [3] for the corresponding deterministic system. The contribution from the stochasticity that modifies the formula is in the second term. An interesting observation is that, if $g_{11}=\Sigma_k b_{1k}^2$ is independent of x_2, this term vanishes. To see this, notice that $\int\rho_{2|1}dx_2=1$, which results in

$$\mathcal{E}_{2|1}\left(\frac{\partial^2(g_{11}\rho_1)}{\partial x_1^2}\right)=\int\frac{\partial^2(g_{11}\rho_1)}{\partial x_1^2}dx_1=0.$$

We thus have the following property:

Given a stochastic system component, if the stochastic perturbation is independent of another component, then the information transfer from the latter is the same in form as that for the corresponding deterministic system.

This property is interesting since a large proportion of noise appearing in real problems is additive, that is to say, b_{ij}, and hence g_{ij}, are often constant. This theorem shows that, in terms of information transfer, these stochastic systems function like deterministic; but, of course, the similarity is just in form; they are different in value. The first part on the right-hand side of Eq. (16) actually has stochasticity embedded in the marginal density. Besides, for deterministic systems the differential entropy may go to minus infinity (think about the attractor of a fixed point or limit cycle [13]), while this does not make a problem for the stochastic case.

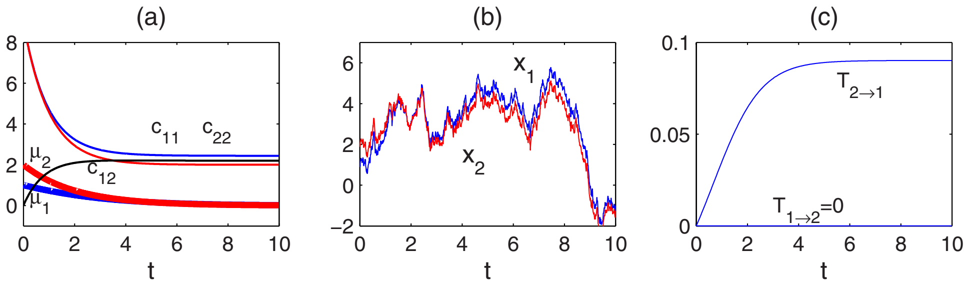

FIG. 1. (Color online) (a) A solution of Eq. (19) with $a_{21}=0$ and initial conditions as shown in the text: $\underline{\mu}$ (thick lines) and $\underline{\underline{\mathbf{C}}}$ (thin lines). (b) A sample path starting from (1,2). (c) The computed information transfers $T_{2\to1}$ (upper) and $T_{1\to2}=0$.

Another property is the concretization of the requirement of transfer asymmetry emphasized in [2]:

If the evolution of x_1 is independent of x_2,

then $T_{2\to1}$ is zero.

In fact, if neither F_1 nor g_{11} have dependency on x_2, the integrals in Eq. (7) can be evaluated and the whole equation becomes a Fokker-Planck equation for ρ_1. In this case, x_1 behaves like an independent variable. So by intuition, there should be no information flowing from x_2. This is indeed true by formula (16). If $F_1=F_1(x_1)$, integration can be made for $\rho_{2|1}$ with respect to x_2 inside the double integral, giving a zero $T_{2\to1}$.

To help further understand the formulas (16) and (17), consider a 2D linear system:

$$d\underline{\mathbf{x}} = \underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}}\,dt + \underline{\underline{\mathbf{B}}}\,d\underline{\mathbf{w}}, \tag{18}$$

where $\underline{\underline{\mathbf{A}}}=(a_{ij})$ and $\underline{\underline{\mathbf{B}}}=(b_{ij})$ are constant matrices. Further suppose that $\underline{\mathbf{x}}$ has an initial Gaussian distribution; it is then Gaussian all the time [14], with a mean $\underline{\mu}=(\mu_1,\mu_2)^T$ and a covariance matrix $\underline{\underline{\mathbf{C}}}=(c_{ij})$ evolving as

$$d\underline{\mu}/dt = \underline{\underline{\mathbf{A}}}\,\underline{\mu}, \tag{19a}$$

$$d\underline{\underline{\mathbf{C}}}/dt = \underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{C}}} + \underline{\underline{\mathbf{C}}}\,\underline{\underline{\mathbf{A}}}^T + \underline{\underline{\mathbf{B}}}\,\underline{\underline{\mathbf{B}}}^T. \tag{19b}$$

The solution of these equations determines the density

$$\rho(\underline{\mathbf{x}}) = \frac{1}{2\pi|\underline{\underline{\mathbf{C}}}|^{1/2}} e^{-1/2(\underline{\mathbf{x}}-\underline{\mu})^T\underline{\underline{\mathbf{C}}}^{-1}(\underline{\mathbf{x}}-\underline{\mu})},$$

which, after substituted into Eqs. (16) and (17), gives the transfers between x_1 and x_2.

For an example, let all the entries of $\underline{\underline{\mathbf{B}}}$ be 1, and $a_{11}=a_{22}=-0.5$, $a_{12}=0.1$, leaving a_{21} open for experiment. First consider $a_{21}=0$. It is easy to show that this system has an equilibrium solution: $\underline{\mu}=(0,0)$, $c_{11}=2.44$, $c_{12}=c_{21}=2.2$, $c_{22}=2$, whatever the initial conditions are. Figure 1(a) shows the time evolutions of $\underline{\mu}$ and $\underline{\underline{\mathbf{C}}}$ initialized with $\underline{\mu}(0)=(1,2)$ and $c_{11}(0)=c_{22}(0)=9$, $c_{12}(0)=c_{21}(0)=0$; also shown is a sample path of $\underline{\mathbf{x}}$ starting from $\underline{\mu}(0)$. In this system, $F_2=-0.5x_2$ has no dependence on x_1, and $g_{ij}=\Sigma_k b_{ik}b_{jk}$ are all constants, so $T_{1\to2}=0$ by the property established above. The computed result confirms this inference. In Fig. 1(c), $T_{1\to2}$ is zero through time. The other transfer, $T_{2\to1}$, increases monotonically and eventually approaches a constant.

An interesting observation about the typical sample path in Fig. 1(b) is the high correlation between x_1 and x_2, in contrast to the zero information transfer $T_{1\to2}$. That is to say, even though $x_1(t)$ and $x_2(t)$ are highly correlated, the evolution of x_2 has nothing to do with x_1. Through this simple example one sees how information transfer may extend the traditional notion of correlation analysis and/or mutual information analysis by including causality [15].

In the second experiment, we let $a_{21}=0.1=a_{12}$, resulting in a system symmetric between x_1 and x_2. One thus naturally expects two transfers equal in value. The computed results show that this is indeed so. The transfer $T_{2\to1}$ is equal to $T_{1\to2}$ (not shown). (If $\mu_1\neq\mu_2$, initially they may be different, but merge together soon after the transient period.) In the third experiment, $a_{21}=0.2>a_{12}$; the influence of x_1 on x_2 is larger than that of x_2 on x_1, so one expects a larger $T_{1\to2}$ than $T_{2\to1}$. Again, the computed result agrees with the inference (not shown). The formulas (16) and (17) are verified with this example.

We have rigorously established a formalism of information transfer within 2D stochastic dynamical systems, which is measured by the rate of entropy transferred from one component to another. The measure possesses a property of transfer asymmetry and, if the stochastic perturbation to the receiving component does not rely on the giving component, has a form the same as that for the corresponding deterministic system. The resulting formulas (16) and (17) can be applied to any systems when the underlying dynamics is given. When the dynamics is unknown but two time series are given, one may fit the data into a 2D model in the form of Eq. (4) and then compute the information transfer. Theoretically this is possible, but technical difficulties still exist in estimating stochastic differential equations. Nonetheless, a large number of problems can be approximately described by linear systems. In that case, an explicit expression of data-based information transfer can be derived from Eqs. (16) and (17). Particularly, for two demeaned time series, $\{x_1^n\}_n$ and $\{x_2^n\}_n$, with stochasticity due to independent perturbations, Eqs. (16) and (17) are reduced to two concise formulas in terms of the covariance matrix between the two series [10]. We have used these formulas to investigate the air-sea interaction that leads to the recent climate change [11] and obtained important results; more applications are expected to follow up in other research fields.

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[1] K. Kaneko, Physica D **23**, 436 (1986); J. A. Vastano and H. L. Swinney, Phys. Rev. Lett. **60**, 1773 (1988); M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, *ibid.* **76**, 1804 (1996); J. Arnhold *et al.*, Physica D **134**, 419 (1999).

[2] T. Schreiber, Phys. Rev. Lett. **85**, 461 (2000); A. Kaiser and T. Schreiber, Physica D **166**, 43 (2002).

[3] X. S. Liang and R. Kleeman, Phys. Rev. Lett. **95**, 244101 (2005).

[4] X. S. Liang and R. Kleeman, Physica D **231**, 1 (2007); **227**, 173 (2007).

[5] A. J. Majda and J. Harlim, Proc. Natl. Acad. Sci. U.S.A. **104**, 9558 (2007).

[6] See E. Pereda *et al.*, Prog. Neurobiol. (Oxford) **77**, 1 (2005), and references therein.

[7] R. Kleeman, J. Atmos. Sci. **64**, 1005 (2007); Physica D **230**, 65 (2007); J. Atmos. Sci. **59**, 2057 (2002).

[8] See, e.g., E. Lorenz, Tellus **17**, 321 (1965); C. E. Leith, Mon. Weather Rev. **102**, 409 (1974); E. S. Epstein, Tellus **21**, 739 (1969); E. Ehrendorfer and J. Tribbia, J. Atmos. Sci. **54**, 286 (1997); T. N. Palmer, Rep. Prog. Phys. **63**, 71 (2000); R. N. Miller and L. L. Ehret, Mon. Weather Rev. **130**, 2313 (2002); Z. Toth and E. Kalnay, *ibid.* **325**, 3297 (1997); A. M. Moore, J. Phys. Oceanogr. **29**, 762 (1999); A. D. Kirwan, M. Toner, and L. Kantha, Int. J. Eng. Sci. **41**, 249 (2003); P. Lermusiaux, J. Comput. Phys. **217**, 176 (2006).

[9] See, e.g., G. F. Carnevale and G. Holloway, J. Fluid Mech. **116**, 115 (1982); B. F. Farrell, J. Atmos. Sci. **47**, 2409 (1990); T. Schneider and S. Griffies, J. Clim. **12**, 3133 (1999); R. Kleeman, J. Atmos. Sci. **59**, 2057 (2002); R. Kleeman, A. Majda, and I. Timofeyev, Proc. Natl. Acad. Sci. U.S.A. **99**, 15291 (2002); A. J. Majda, R. Kleeman, and D. Cai, Methods Appl. Anal. **9**, 425 (2002); R. Kleeman and A. Majda, J. Atmos. Sci. **62**, 2864 (2005); R. Kleeman, *ibid.* **64**, 1005 (2007); Physica D **230**, 65 (2007); R. Abramov, A. Majda, and R. Kleeman, J. Atmos. Sci. **62**, 65 (2005).

[10] X. S. Liang (unpublished).

[11] X. S. Liang (unpublished).

[12] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise* (Springer, New York, 1994).

[13] D. Ruelle, J. Stat. Phys. **85**, 1 (1996); **86**, 935 (1997).

[14] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences* (Springer-Verlag, Berlin, 1985).

[15] C. Granger, Econometrica **37**, 424 (1969), first introduced the concept of causality. Also see C. Sims, *ibid.* **48**, 1 (1980); H. Toda and P. Philips, *ibid.* **61**, 1367 (1993); K. Hlaváčková-Schindler *et al.*, Phys. Rep. **441**, 1 (2007).