Canonical Transfer and Multiscale Energetics for Primitive and Quasigeostrophic Atmospheres

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ABSTRACT

The past years have seen the success of a novel and rigorous localized multiscale energetics formalism in a variety of ocean and engineering fluid applications. In a self-contained way, this study introduces it to the atmospheric dynamical diagnostics, with important theoretical updates and clarifications of some common misconceptions about multiscale energy. Multiscale equations are derived using a new analysis apparatus—namely, multiscale window transform—with respect to both the primitive equation and quasigeostrophic models. A reconstruction of the "atomic" energy fluxes on the multiple scale windows allows for a natural and unique separation of the in-scale transports and cross-scale transfers from the intertwined nonlinear processes. The resulting energy transfers bear a Lie bracket form, reminiscent of the Poisson bracket in Hamiltonian mechanics; hence, we would call them "canonical." A canonical transfer process is a mere redistribution of energy among scale windows, without generating or destroying energy as a whole. By classification, a multiscale energetic cycle comprises available potential energy (APE) transport, kinetic energy (KE) transport, pressure work, buoyancy conversion, work done by external forcing and friction, and the cross-scale canonical transfers of APE and KE, which correspond respectively to the baroclinic and barotropic instabilities in geophysical fluid dynamics. A buoyancy conversion takes place in an individual window only, bridging the two types of energy, namely, KE and APE; it does not involve any processes among different scale windows and is hence basically not related to instabilities. This formalism is exemplified with a preliminary application to the study of the Madden–Julian oscillation.

1. Introduction

Ever since Lorenz (1955) introduced the concept of available potential energy (APE), and set up a two-scale formalism of energy equations using the Reynolds decomposition, energetic analysis has become a powerful tool for diagnosing atmospheric and oceanic processes. Related studies include mean flow–wave interaction (e.g., Dickinson 1969; Boyd 1976; McWilliams and Restrepo 1999; Fels and Lindzen 1974; Matsuno 1971), upward propagation of planetary-scale disturbances (Charney and Drazin 1961), ocean circulation energetics (Holland 1978; Haidvogel et al. 1992), mean current–eddy interaction (Hoskins et al. 1983), atmospheric

blocking (Trenberth 1986; Fournier 2002; Luo et al. 2014), Gulf Stream dynamics (Dewar and Bane 1989), normal modal interaction (Sheng and Hayashi 1990), regional cyclogenesis (Cai and Mak 1990), convection and cabbeling (Su et al. 2016), and the most recent studies such as Cai et al. (2007), Waterman and Jayne (2011), Murakami (2011), Hsu et al. (2011), Chen et al. (2014), and Chapman et al. (2015), to name a few. Meanwhile, Saltzman (1957) cast the problem into the framework of Fourier analysis and obtained the energetics in the wavenumber domain, while Kao (1968) further extended it to the wavenumber–frequency space. Now both approaches have become standard in geophysical fluid dynamics and other fluid-related fields; see, for example, Pedlosky (1987), Chorin (1994), and Pope (2004).

Lorenz's energetics in bulk form (i.e., in the form of global mean or integral) have clear physical interpretations (e.g., Pedlosky 1987). This global-mean form, however, may be inappropriate for regional diagnostics, as real atmospheric processes are localized in nature; in other words, they tend to be locally defined in

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space and time and can be on the move. The Madden–Julian oscillation (MJO) that we will take a brief look at the end of this study is such an example; it is a progressive process that involves energy production and dissipation. For this reason, it has been a continuing effort to relax the spatial averaging/integration to have these processes faithfully represented. A tradition started by Lorenz himself is to collect the terms in divergence form and combine them as one term representing the transport process, separate the term from the nonlinear interaction, and take the residue as the energy transfer between the distinct scales (e.g., Harrison and Robinson 1978). Now this has been a standard approach to multiscale energetic diagnostics for fluid research, particularly for turbulence research, where much effort has been devoted to engineering the so-obtained transfers (cf. Pope 2004).

While we know a transport process indeed bears a divergence form in the governing equations, the above transport–transfer separation is not unique. Multiple divergence forms exist that may yield quite different transfers. As argued by Holopainen (1978), the resulting energy transfer in such an open system is quite ambiguous. This issue, which is actually quite profound in fluid dynamics, has long been discovered but has not received enough attention, except for a few studies such as Plumb (1983). [The discussion by Berloff (2005) on the consistency of eddy fluxes also seems to be related to this problem.]

Another major issue in formulating multiscale energetics regards the machinery for process decomposition by scale. Traditionally, two methods—namely, Reynolds' mean eddy decomposition (MED) and Fourier transform—have been used. The former is originally a statistical notion with respect to an ensemble mean, but for practical reasons the ensemble mean is usually replaced by time mean, zonal mean, etc., making it a tool of scale decomposition. Both of these methods are global, in the sense that they do not retain the local information. This is generally inappropriate for realistic atmospheric processes such as instabilities, which are in nature highly localized energy burst processes. In remedy, a practical approach that is commonly used is to do a running time mean over a chosen period of time. Indeed, this gives the local information while retaining the simplicity of the Reynolds formalism. However, it does not solve the fundamental problem that an energy burst process, among others, is by no means stationary over any duration; any scale decomposition under such a hidden assumption may result in spurious information, preventing one from making correct diagnoses.

An alternative approach to overcoming the difficulty is via filtering. Filters have been widely used to separate

processes involving different scales. But for energetics studies, it seems that a very fundamental issue has been completely ignored—that is, how energy (and any quadratic properties) should be expressed in this framework. Currently the common practice is, for a two-scale decomposition, to first apply some filter to separate a field variable, say, u, into two parts, say, u_L and u_S, which represent the large-scale and small-scale features, respectively, and then take u_L^2 and u_S^2 (up to some factor) as the large-scale and small-scale energies. While this intuitively based and widely used technique may be of some use in real problem diagnostics, it is not physically relevant—one immediately sees the inadequacy by noticing that $u^2 \neq u_L^2 + u_S^2$. In fact, multiscale energy is a concept in phase space, such as that in Fourier power spectra; it is related to physical energy through a theorem called the Parseval relation. Attempting to evaluate multiscale energies with the filtered (low pass, bandpass, etc.) or reconstructed field variables is conceptually off track. Actually this is a difficult problem and has not been well formulated until filter banks and wavelets are connected (Strang and Nguyen 1997). Besides, energy conservation requires that the resulting subspaces from filtering must be orthogonal, as we will elaborate in the following section. This requirement, unfortunately, has been mostly ignored in previous studies along this line.

The other line in this regard is with respect to Fourier transform [Saltzman (1957) and its sequels], which does not have local information retained, either. Coming to remedy is wavelet transform or, to be precise, orthonormal wavelet transform (OWT), as only with an orthogonal basis can the notion of energy in the physical sense be introduced. OWT was first introduced by Fournier (2002) into the study of atmospheric energetics. This is a formalism with respect to space. While opening a door to localized spectral structures, many processes such as transports are not as easy to see as those in the Lorenz-type formalisms. On the other hand, the atmospheric and oceanic processes tend to occur on a range of scales (e.g., MJO has a scale range of 30–60 days), or "scale windows" as we will introduce in the following section, rather than on individual scales. For OWT, transform coefficients (hence, multiscale energies) are defined discretely at different locations for different scale levels; there is no way to add them through a range of scales to make an expression of localized energy for that range. These issues, among others, are yet to be addressed with these formalisms.

So, to relax the spatial averaging in a bulk energetics formalism incurs the issue of transport–transfer separation, while improving MED to have local information retained requires more sophisticated machinery of scale decomposition. Can we put these two issues in the same

framework and solve them in a unified approach? The answer is yes. The early attempts include the multiscale oceanic energetics studies by Liang and Robinson (2005, hereafter LR05), based on multiscale window transform (MWT), a functional analysis tool that was rigorized later (Liang and Anderson 2007, hereafter LA07). This formalism has been mostly overlooked, though it has been applied with success to a variety of real ocean problems (e.g., Liang and Robinson 2004, 2009) and engineering problems (e.g., Liang and Wang 2004), partly because it has not been introduced for atmospheric studies and has not been formulated in spherical coordinates. (As we will see soon, expressing the energetics in spherical coordinates is by no means an easy task.) This study is purported to address these issues, giving a comprehensive and self-contained introduction of the fundamentals and the progress since LR05. A key point that distinguishes this study from the earlier effort is that, in LR05, the transport–transfer separation was introduced in a half-empirical way. With the nice properties of the MWT, which was formally established later on in LA07, we will see soon in the next section that this actually can be put on a rigorous footing, and the resulting transfer bears a Lie bracket form, reminding us of the Poisson bracket in Hamiltonian mechanics. Besides, in this study we will extend the formalism to quasigeostrophic flows, which must be derived in a different way. Considering the traditional and recently renewed interests (e.g., Murakami 2011) in multiscale atmospheric energetics diagnostics, and considering that a topic of much concern in turbulence research is to engineer the resulting transfer, this rigorous study is rather timely.

In the following we first give a brief introduction of the concepts of scale window, multiscale window transform, and multiscale energy. In section 3, we show how the flux on a specific scale window can be rigorously derived and how the energy transfer between two-scale windows can be obtained. We will see that the resulting transfer bears a form like the Lie bracket, reminding one of the Poisson bracket in Hamiltonian dynamics. We then derive the evolution equations for the multiscale kinetic energy (KE) and APE with both a primitive atmospheric model (section 4) and a quasigeostrophic model (section 6). For completeness, a summary of the multiscale oceanic energetics, together with the needed modification, is briefly presented (section 5); also included is a brief review of some necessary horizontal treatment (section 7). In section 8, we demonstrate how the formalism may be applied, using the Madden–Julian oscillation as an example. This study is summarized in section 9. For easy reference, in appendix A, a glossary of symbols is provided. The related software can be downloaded from the website http://www.ncoads.org/ (within the section "software").

2. Multiscale window transform

This section gives a very brief introduction to the multiscale window transform developed by LA07. The first part (section 2a) is the fundamentals, but the reader may simply skip ahead if he/she already knows the notation and the fact that a reconstruction is conceptually different from a transform.

a. Scale window and multiscale window transform

More often than not, an atmospheric process tends to occur on a range of scales, such as the MJO, which has a broadband spectrum between 30 and 60 days (cf. section 8), rather than on individual scales. Such a scale range is called, in a loose sense, a scale window. Rigorously it can be defined over a univariate interval or a multidimensional domain. In this study, the former is used, as we only deal with time. This is in accordance with Lorenz's formalism. Historically it has long been discussed (e.g., Haynes 1988) and has been justified by the observational fact that, in the atmosphere, scales in time and in space are correlated. Besides, only scales defined over a univariate field can be unambiguously referred to as large scale, small scale, and so forth, as desired in the atmospheric energetics studies.

Without loss of generality, let the interval over which the signals to be diagnosed span be $[0, 1]$; if not, it may always be made so after a transformation. Consider a Hilbert space $V_{\ell,j} \subset L_2[0, 1]^1$ generated by the basis $\{\phi_n^j(t)\}_{n=0,1,\dots,2^j\ell-1}$, where

$$\phi_n^j(t) = \sum_{q=-\infty}^{+\infty} 2^{j/2}\phi[2^j(t + \ell q) - n + 1/2],$$
$$n = 0, 1, \dots, 2^{j\ell-1}. \qquad (1)$$

Here $\phi(t)$ is a scaling function constructed in LA07 such that $\{\phi(t - n + 1/2)\}_n$ is orthonormal (Fig. 1). From $\phi(t)$, one can also construct an orthonormal wavelet basis. The parameter $\ell = 1$ or $\ell = 2$, corresponding respectively to the periodic and symmetric extension schemes. Shown in Fig. 2 is the basis for $\ell = 2$ and a selection of j—namely, the "scale level" (2^{-j} is the scale). For notational simplicity, throughout this study the dependence of ϕ_n^j on ℓ is suppressed (but retained in other notations).

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[1] Loosely speaking, it is a space of square integrable functions on $[0, 1]$.
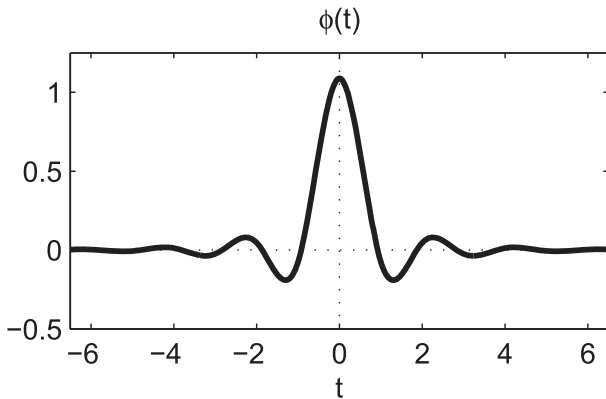
FIG. 1. The orthonormal scaling function ϕ constructed in LA07.

It has been justified in LA07 that there always exists a j_2 such that all the atmospheric/oceanic signals of concern lie in V_{ℓ,j_2}. Furthermore, it has been shown in there that

$$V_{\ell_{j_0}} \subset V_{\ell_{j_1}} \subset V_{\ell_{j_2}}, \quad \text{for} \quad j_0 < j_1 < j_2.$$

A decomposition thus can be made such that

$$V_{\ell_{j_2}} = V_{\ell_{j_1}} \oplus W_{\ell_{j_1-j_2}} = V_{\ell_{j_0}} \oplus W_{\ell_{j_0-j_1}} \oplus W_{\ell_{j_1-j_2}}, \quad (2)$$

where $W_{\ell_{j_1-j_2}}$ is the orthogonal complement of $V_{\ell_{j_1}}$ in $V_{\ell_{j_2}}$, and $W_{\ell_{j_0-j_1}}$ that of $V_{\ell_{j_0}}$ in $V_{\ell_{j_1}}$. It has been shown by LA07 that $V_{\ell_{j_0}}$ contains functions of scales larger than 2^{-j_0} only, while lying in $W_{\ell_{j_0-j_1}}$ and $W_{\ell_{j_1-j_2}}$ are the functions with scale ranges from 2^{-j_0} to 2^{-j_1} and from 2^{-j_1} to 2^{-j_2}, respectively. We call the so-formed subspaces of $V_{\ell_{j_2}}$ as scale windows. For easy reference, from larger scales (lower scale levels) to smaller scales (higher scale levels), they will be referred to as scale windows 0, 1, and 2, respectively. Depending on the problem of concern, they may also be assigned names in association to physical processes. For example, one may refer to them as large-scale, midscale, and small-scale windows, or, in the context of, say, MJO studies, mean window, intraseasonal window or MJO window, and synoptic window, or, in the context of oceanography, large-scale window, mesoscale window, and submesoscale window. More scale windows can be likewise defined, but in this study, usually three are enough (in fact, in many cases only two are needed).

Consider a function $u(t) \in V_{\ell_{j_2}}$. With (1), a transform

$$\hat{u}_n^j = \int_0^\ell u(t)\phi_n^j(t)\,dt \quad (3)$$

can be defined for a scale level j. Given window bounds $j_0 < j_1 < j_2$, u then can be reconstructed on the three scale windows as constructed above:

$$u^{\sim 0}(t) = \sum_{n=0}^{2^{j_0}\ell-1} \hat{u}_n^{j_0}\phi_n^{j_0}(t), \quad (4)$$

$$u^{\sim 1}(t) = \sum_{n=0}^{2^{j_1}\ell-1} \hat{u}_n^{j_1}\phi_n^{j_1}(t) - u^{\sim 0}(t), \quad \text{and} \quad (5)$$

$$u^{\sim 2}(t) = u(t) - u^{\sim 0}(t) - u^{\sim 1}(t), \quad (6)$$

with the notations ~ 0, ~ 1, and ~ 2 signifying, respectively, the corresponding three scale windows. Since $V_{\ell_{j_0}}$, $W_{\ell_{j_0-j_1}}$, and $W_{\ell_{j_1-j_2}}$ are all subspaces of $V_{\ell_{j_2}}$, the functions $u^{\sim 0}$, $u^{\sim 1}$, and $u^{\sim 2}$ can be transformed with respect to $\{\phi_n^{j_2}(t)\}_n$, the basis of $V_{\ell_{j_2}}$,

$$\hat{u}_n^{\sim \varpi} = \int_0^\ell u^{\sim \varpi}(t)\phi_n^{j_2}(t)\,dt, \quad (7)$$

for windows $\varpi = 0, 1, 2$, and $n = 0, 1, \ldots, 2^{j_2}\ell - 1$. Note here the transform coefficients $\hat{u}_n^{\sim \varpi}$ contain only the processes belonging to scale window ϖ. It has, though discretely, the finest resolution permissible in the sampling space on [0, 1]. We call (7) a multiscale window transform. With this, (4)–(6) can be written in a unified way:

$$u^{\sim \varpi}(t) = \sum_{n=0}^{2^{j_2}\ell-1} \hat{u}_n^{\sim \varpi}\phi_n^{j_2}(t), \quad \varpi = 0, 1, 2. \quad (8)$$

Equations (7) and (8) form the transform–reconstruction pair for MWT.

b. Multiscale energy

MWT has a Parseval relation–like property; in the periodical extension case ($\ell = 1$),

$$\sum_n \hat{u}_n^{\sim \varpi}\hat{v}_n^{\sim \varpi} = \overline{u^{\sim \varpi}(t)v^{\sim \varpi}(t)}, \quad (9)$$

for $u, v \in V_{1,j_2}$, and because of the mutual orthogonality between the scale windows,

$$\sum_\varpi \sum_n \hat{u}_n^{\sim \varpi}\hat{v}_n^{\sim \varpi} = \overline{u(t)v(t)}, \quad (10)$$

where the overline indicates averaging over time, and $\sum_n$ is a summation over the sampling set $\{0, 1, 2, \ldots, 2^{j_2} - 1\}$ (see LA07 for a proof). In the case of other extensions, $\sum_n$ is replaced by "marginalization," a naming convention after Huang et al. (1999), which also bears the physical meaning of summation over n. Equation (10) states that, a product of two MWT coefficients followed by a marginalization is equal to the product of their corresponding reconstructions averaged over the duration. This property is usually referred to as property of marginalization.

The property of marginalization is important in that it allows for an efficient representation of multiscale energy in terms of the MWT transform coefficients. In (10), let $u = v$, the right-hand side is then the energy of u
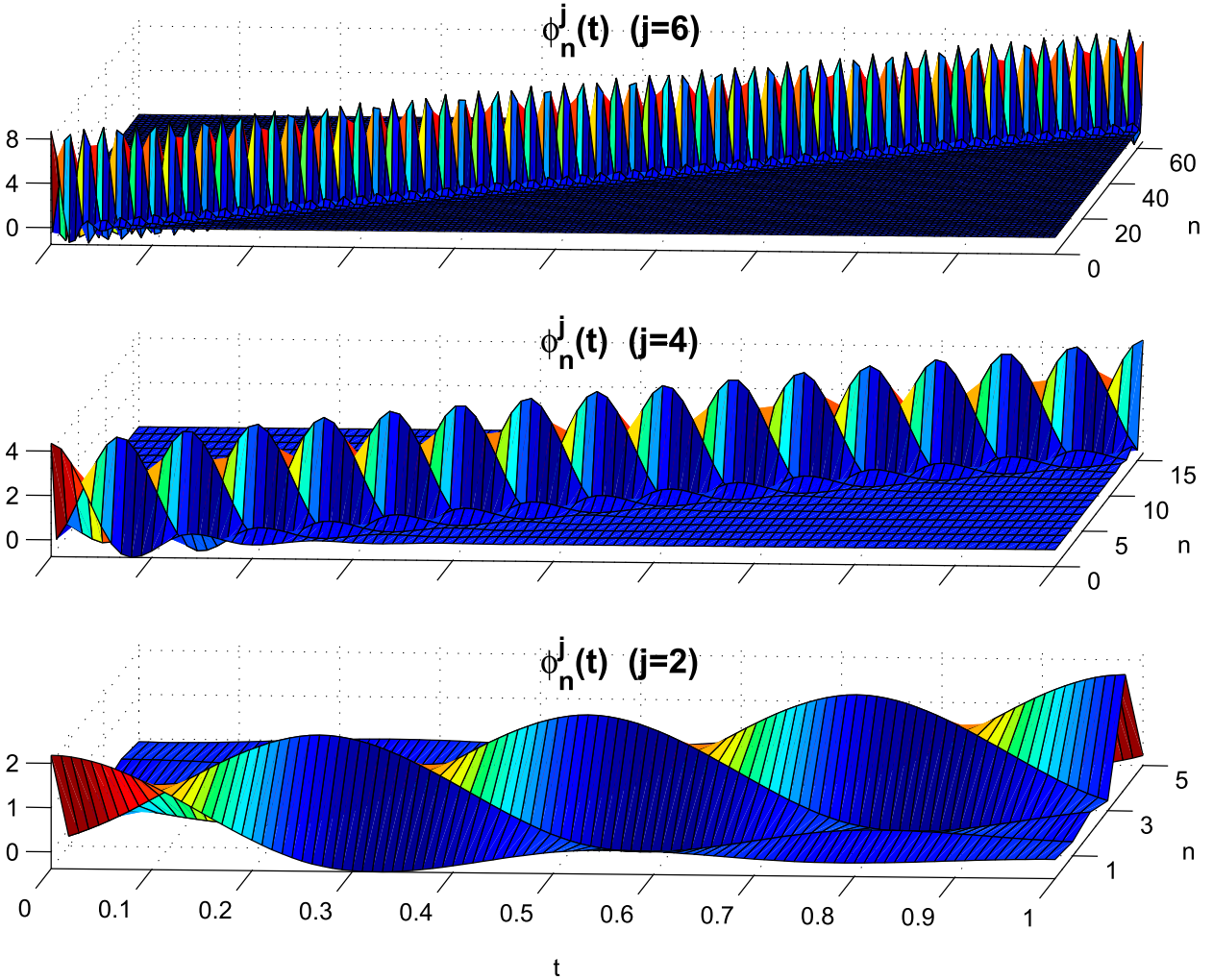
FIG. 2. Three-dimensional plot of ϕ_n^j as function of t (physical space variable; here time) and n (sampling space variable) for a selection of scale level j (corresponding to scale 2^{-j} on a $[0, 1]$ domain). The symmetric extension scheme is used in constructing ϕ_n^j via the ϕ in Fig. 1.

(up to some constant factor) averaged over $[0, 1]$. It is equal to a summation of $3N = 3 \times 2^{j_2}$ (if three scale windows are considered) individual objects $(\hat{u}_n^{\sim \varpi})^2$ centered at time $t_n = 2^{-j_2}n + 1/2$, with a characteristic influence interval $\Delta t = t_{n+1} - t_n = 2^{-j_2}$. The multiscale energy at time t_n then should be the mean over the interval: $(\hat{u}_n^{\sim \varpi})^2/\Delta t = 2^{j_2}(\hat{u}_n^{\sim \varpi})^2$. Notice the constant multiplier 2^{j_2}; it is needed for the obtained multiscale energy to make sense in physics. But for notational succinctness, it will be omitted in the following derivations.

Therefore, the energy of u on scale window ϖ at step n is

$$E_n^\varpi \propto (\hat{u}_n^{\sim \varpi})^2. \tag{11}$$

Note the ϖ-window-filtered signal is $u^{\sim \varpi}$; by common practice, one would take $(u^{\sim \varpi})^2$ as the energy on ϖ. From above, one sees that this is conceptually incorrect.

3. Multiscale flux and canonical transfer

a. Multiscale flux

For a scalar field T, its "energy" (quadratic property) on window ϖ at step n is $(\hat{T}_n^{\sim \varpi})^2/2$ (up to some factor). In the MWT framework, energy can be decomposed as a sum of a bunch of atomlike elements:

$$\frac{1}{2}T^2 = \sum_{n_1, \varpi_1} \sum_{n_2, \varpi_2} \frac{1}{2}[\hat{T}_{n_1}^{\sim \varpi_1}\phi_{n_1}^{j_2}(t)][\hat{T}_{n_2}^{\sim \varpi_2}\phi_{n_2}^{j_2}(t)]. \tag{12}$$

Look at the flux of the "atom" by a flow $\mathbf{v}(t)$ over $t \in [0, 1]$ at step n within window ϖ. It is

$$\int_0^1 \mathbf{v}(t) \cdot \frac{1}{2}[\hat{T}_{n_1}^{\sim \varpi_1}\phi_{n_1}^{j_2}(t)][\hat{T}_{n_2}^{\sim \varpi_2}\phi_{n_2}^{j_2}(t)]\delta(n - n_2)\delta(\varpi - \varpi_2)\, dt. \tag{13}$$

In the above delta functions, the arguments may equally be chosen as n_1 and ϖ_1. The flux of $T^2/2$ by the flow $\mathbf{v}$ on ϖ at step n is then the sum of the atomic expressions over all the possible n_1, n_2, ϖ_1, and ϖ_2; that is,

$$
\begin{aligned}
\mathbf{Q}_n^\varpi &= \sum_{n_1,\varpi_1} \sum_{n_2,\varpi_2} \int_0^1 \frac{1}{2} \mathbf{v} \cdot [\hat{T}_{n_1}^{\sim\varpi_1} \phi_{n_1}^{j_2}(t)][\hat{T}_{n_2}^{\sim\varpi_2} \phi_{n_2}^{j_2}(t)] \delta(n-n_2)\delta(\varpi-\varpi_2)\, dt \\
&= \frac{1}{2} \int_0^1 \mathbf{v}(t) T(t) \cdot \hat{T}_n^{\sim\varpi} \phi_n^{j_2}(t)\, dt.
\end{aligned}
\tag{14}
$$

But the function $\hat{T}_n^{\sim\varpi} \phi_n^{j_2}(t)$ lies in window ϖ, and all windows are orthogonal, so this is something like a projection of $\mathbf{v}T$ onto window ϖ:

$$
\begin{aligned}
\mathbf{Q}_n^\varpi &= \frac{1}{2} \int_0^1 \widehat{(\mathbf{v}T)}_n^{\sim\varpi} \cdot \hat{T}_n^{\sim\varpi} \phi_n^{j_2}(t)\, dt \\
&= \frac{1}{2} \hat{T}_n^{\sim\varpi} \widehat{(\mathbf{v}T)}_n^{\sim\varpi}.
\end{aligned}
\tag{15}
$$

The above equation can be used for the derivation of multiscale potential energetics. For kinetic energy $K = (1/2)\mathbf{v} \cdot \mathbf{v}$, essentially one can derive in the same way. To avoid confusion, we consider the energylike quantity of an arbitrary vector $\mathbf{G}$,

$$
K = \frac{1}{2}\mathbf{G} \cdot \mathbf{G} = \sum_{n_1,\varpi_1} \sum_{n_2,\varpi_2} \frac{1}{2}[\hat{\mathbf{G}}_{n_1}^{\sim\varpi_1} \phi_{n_1}^{j_2}(t)] \cdot [\hat{\mathbf{G}}_{n_2}^{\sim\varpi_2} \phi_{n_2}^{j_2}(t)].
\tag{16}
$$

So the flux of the atom over $t \in [0, 1]$ at step n on window ϖ is

$$
\int_0^1 \mathbf{v}(t) \frac{1}{2}[\hat{\mathbf{G}}_{n_1}^{\sim\varpi_1} \phi_{n_1}^{j_2}(t)] \cdot [\hat{\mathbf{G}}_{n_2}^{\sim\varpi_2} \phi_{n_2}^{j_2}(t)] \delta(n-n_2)\delta(\varpi-\varpi_2)\, dt,
\tag{17}
$$

and the flux of K by $\mathbf{v}$ on ϖ at n is

$$
\begin{aligned}
\mathbf{Q}_n^\varpi &= \sum_{n_1,\varpi_1} \sum_{n_2,\varpi_2} \frac{1}{2} \mathbf{v}(t)[\hat{\mathbf{G}}_{n_1}^{\sim\varpi_1} \phi_{n_1}^{j_2}(t)] \\
&\quad \cdot [\hat{\mathbf{G}}_{n_2}^{\sim\varpi_2} \phi_{n_2}^{j_2}(t)] \delta(n-n_2)\delta(\varpi-\varpi_2)\, dt \\
&= \frac{1}{2} \int_0^1 [\mathbf{v}(t)\mathbf{G}(t)] \cdot \hat{\mathbf{G}}_n^{\sim\varpi} \phi_n^{j_2}(t)\, dt,
\end{aligned}
\tag{18}
$$

where the dyadic $\mathbf{v}\mathbf{G}$ takes right dot product with $\hat{\mathbf{G}}_n^{\sim\varpi}$. Again, $\hat{\mathbf{G}}_n^{\sim\varpi} \phi_n^{j_2}(t)$ lies in window ϖ. Because of the orthogonality among windows,

$$
\begin{aligned}
\mathbf{Q}_n^\varpi &= \frac{1}{2} \int_0^1 [\mathbf{v}(t)\mathbf{G}(t)]^{\sim\varpi} \cdot \hat{\mathbf{G}}_n^{\sim\varpi} \phi_n^{j_2}(t)\, dt \\
&= \frac{1}{2} \widehat{(\mathbf{v}\mathbf{G})}_n^{\sim\varpi} \cdot \hat{\mathbf{G}}_n^{\sim\varpi} \\
&= \frac{1}{2}[\widehat{(\mathbf{v}G_1)}_n^{\sim\varpi} \widehat{(G_1)}_n^{\sim\varpi} + \widehat{(\mathbf{v}G_2)}_n^{\sim\varpi} \widehat{(G_2)}_n^{\sim\varpi}],
\end{aligned}
\tag{19}
$$

which is like the superposition of the fluxes of two scalar fields—namely, G_1 and G_2.

b. Canonical transfer

Consider a scalar property T in an incompressible flow field $\mathbf{v}$. The equation governing the evolution of T is

$$
\frac{\partial T}{\partial t} + \nabla \cdot (\mathbf{v}T) = \text{other terms}.
$$

As only the nonlinear term—namely, the advection—will lead to interscale transfer, all other terms (e.g., diffusion, source/sink) are unexpressed and put to the right-hand side. To find its evolution on window ϖ, take MWT on both sides. The first term is $(\partial \widehat{T}/\partial t)_n^{\sim\varpi}$. It has been shown by LR05 to be approximately equal to $\delta \hat{T}_n^{\sim\varpi}/\delta n$, where $\delta/\delta n$ is the difference operator with respect to n. Since t of the physical space is now carried over to n of the sampling space, the difference operator is essentially the time rate of change when applying to a discrete time series. We therefore would write it as $\partial \hat{T}_n^{\sim\varpi}/\partial t$ to avoid introducing extra notations, which are already too many. But the careful reader should bear in mind that here it means the difference in the sampling space rather than the differential in the physical space. (Since the signals are sampled at each time step, in real applications they are precisely the same.) The MWTed equation is, therefore,

$$
\frac{\partial \hat{T}_n^{\sim\varpi}}{\partial t} + \nabla \cdot \widehat{(\mathbf{v}T)}_n^{\sim\varpi} = \dots .
$$

Multiplication of $\hat{T}_n^{\sim\varpi}$ gives

$$
\frac{\partial E_n^\varpi}{\partial t} = -\hat{T}_n^{\sim\varpi} \nabla \cdot \widehat{(\mathbf{v}T)}_n^{\sim\varpi} + \dots ,
\tag{20}
$$

where $E_n^\varpi = (\hat{T}_n^{\sim\varpi})^2/2$ is the energy on window ϖ at step n.

One continuing effort in multiscale energetics study is to separate $-\hat{T}_n^{\sim\varpi} \nabla \cdot \widehat{(\mathbf{v}T)}_n^{\sim\varpi}$ into a transport process term $\nabla \cdot \mathbf{Q}_n^\varpi$ and a transfer process term Γ_n^ϖ. Symbolically, this is

$$-\nabla \cdot \mathbf{Q}_n^\varpi + \Gamma_n^\varpi.$$

An intuitively and empirically based common practice is to collect divergence terms to form the transport term (e.g., Harrison and Robinson 1978; Pope 2004). However, as long pointed by people such as Holopainen (1978) and Plumb (1983), among others, there exist other forms that may result in different separations.

In this study, the separation is natural. The multiscale flux $\mathbf{Q}_n^\varpi$, hence the multiscale transport, has been rigorously obtained in the preceding subsection [i.e., (15)]. The transfer Γ is obtained by subtracting $-\nabla \cdot \mathbf{Q}_n^\varpi$ from the right-hand side of (20):

$$\Gamma_n^\varpi = -\hat{T}_n^{\sim\varpi}\nabla \cdot \widehat{(\mathbf{v}T)}_n^{\sim\varpi} + \nabla \cdot \left[\frac{1}{2}\hat{T}_n^{\sim\varpi}\widehat{(\mathbf{v}T)}_n^{\sim\varpi}\right]$$

$$= \frac{1}{2}[\widehat{(\mathbf{v}T)}_n^{\sim\varpi} \cdot \nabla \hat{T}_n^{\sim\varpi} - \hat{T}_n^{\sim\varpi}\nabla \cdot \widehat{(\mathbf{v}T)}_n^{\sim\varpi}]. \qquad (21)$$

Notice that the resulting transfer bears a form similar to the Lie bracket and, particularly, the Poisson bracket in Hamiltonian mechanics. To see this, recall that a Poisson bracket $\{\cdot, \cdot\}$ is defined, for differential operators $(\partial/\partial x, \partial/\partial y)$ and functions F and G, such that

$$\{F, G\} = \frac{\partial F}{\partial y}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial y}.$$

If $\{F, G\} = 0$, then F and G are said to be in involution or to Poisson commute. Consider the 1D version of Γ_n^ϖ; that is,

$$\frac{1}{2}\left[\widehat{(uT)}_n^{\sim\varpi}\frac{\partial \hat{T}_n^{\sim\varpi}}{\partial x} - \hat{T}_n^{\sim\varpi}\frac{\partial \widehat{(uT)}_n^{\sim\varpi}}{\partial x}\right].$$

If we pick two differential operators $(\partial/\partial x, \mathbb{I})$, where $\mathbb{I}$ is the identity, then the above canonical transfer is simply $\{\widehat{(uT)}_n^{\sim\varpi}, \hat{T}_n^{\sim\varpi}\}/2$. Because of this, we will refer it to as canonical transfer in the future, in order to distinguish it from other transfers already existing in the literature.

Canonical transfers possess a very important property, as stated in the following theorem:

Theorem 3.1: A canonical transfer vanishes upon summation over all the scale windows and marginalization over the sampling space; that is,

$$\sum_n \sum_\varpi \Gamma_n^\varpi = 0. \qquad (22)$$

Remark: This theorem states that a canonical transfer process only redistributes energy among scale windows, without generating or destroying energy as a whole. This is precisely that one would expect for an energy transfer process. This property, though natural, generally does not hold for the existing empirical formalisms.

Proof: By the property of marginalization [(9)], (21) gives

$$\sum_n \Gamma_n^\varpi = \frac{1}{2}\int_0^1 [(\mathbf{v}T)^{\sim\varpi} \cdot \nabla T^{\sim\varpi} - T^{\sim\varpi}\nabla \cdot (\mathbf{v}T)^{\sim\varpi}]\, dt.$$

Because of the orthogonality between different scale windows, this followed by a summation over ϖ results in

$$\frac{1}{2}\int_0^1 [(\mathbf{v}T) \cdot \nabla T - T\nabla \cdot (\mathbf{v}T)]\, dt = 0.$$

In the above derivation, the incompressibility assumption of the flow has been used.

The canonical transfer [(21)] may be further simplified in expression when $\hat{T}_n^{\sim\varpi}$ is nonzero:

$$\Gamma_n^\varpi = -E_n^\varpi\nabla \cdot \left[\frac{\widehat{(\mathbf{v}T)}_n^{\sim\varpi}}{\hat{T}_n^{\sim\varpi}}\right], \quad \text{if} \quad \hat{T}_n^{\sim\varpi} \neq 0, \qquad (23)$$

where $E_n^\varpi = (\hat{T}_n^{\sim\varpi})^2/2$ is the energy on window ϖ at step n and is, hence, always positive. Note that (23) defines a field variable which has the dimension of velocity in physical space:

$$\mathbf{v}_T^\varpi = \frac{\widehat{(T\mathbf{v})}_n^{\sim\varpi}}{\hat{T}_n^{\sim\varpi}}. \qquad (24)$$

It may be loosely understood as a weighted average of $\mathbf{v}$, with the weights derived from the MWT of the scalar field T. For convenience, we will refer to $\mathbf{v}_T^\varpi$ as T-coupled velocity. The growth rate of energy on window ϖ is now totally determined by $-\nabla \cdot \mathbf{v}_T^\varpi$, the convergence of $\mathbf{v}_T^\varpi$, and

$$\Gamma_n^\varpi = -E_n^\varpi\nabla \cdot \mathbf{v}_T^\varpi. \qquad (25)$$

Note Γ_n^ϖ makes sense even when $\hat{T}_n^{\sim\varpi} = 0$ and hence $\mathbf{v}_T^\varpi$ does not exist. In this case, (25) should be understood as (21).

The canonical transfer has been validated in many applications. Particularly, it verifies the barotropic instability structure of the Kuo jet stream model, which fails the classical empirical formalism. To facilitate the comparison, Liang and Robinson (2007) established that, when $j_0 = 0$ and a periodical extension is used, the canonical transform [(21)] is reduced to

$$\frac{1}{2}[\overline{T}\nabla \cdot \overline{\mathbf{v}'T'} - \overline{\mathbf{v}'T'} \cdot \nabla\overline{T}]$$

(the overbar indicates a time mean over the whole duration), which is also in a Lie bracket form. This is quite different from the traditional transfer $-\overline{\mathbf{v}'T'} \cdot \nabla \overline{T}$, which, when T is a component of velocity, is usually understood as the energy extracted by the Reynolds stress against the basic profile $\overline{T}$. As demonstrated in Liang and Robinson (2007), this "Reynolds extraction" does not verify the analytical solution of the Kuo instability model, while our canonical transfer does.

4. Multiscale atmospheric energetics

We now apply the above theory to derive the multiscale atmospheric energetics. For notational brevity, from now on the dependence on n will be suppressed in the MWT terms, unless otherwise indicated.

a. Primitive equations

Consider an ideal gas and assume hydrostaticity to hold. We adopt an isobaric coordinate system, which is advantageous over others in that air may be viewed as incompressible, and, besides, as we will see, the resulting energy equations are free of density. The governing equations are (e.g., Salby 1996)

$$\frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h \cdot \nabla_h \mathbf{v}_h + \omega \frac{\partial \mathbf{v}_h}{\partial p} + f\mathbf{k} \times \mathbf{v_h} = -\nabla_h \Phi^* + \mathbf{F}_{m,p} + \mathbf{F}_{m,h},$$
(26)

$$\frac{\partial \Phi^*}{\partial p} = -\alpha^*,$$
(27)

$$\nabla_h \cdot \mathbf{v}_h + \frac{\partial \omega}{\partial p} = 0,$$
(28)

$$\frac{\partial T^*}{\partial t} + \mathbf{v}_h \cdot \nabla_h T^* + \omega \frac{\partial T^*}{\partial p} - \frac{\alpha^* \omega}{c_p} = \frac{\dot{q}_{\text{net}}}{c_p}, \quad \text{and}$$
(29)

$$p\alpha^* = RT^*,$$
(30)

where $\dot{q}_{\text{net}}$ stands for the heating rate from all diabatic sources, $\omega = dp/dt$, the variables with asterisks mean the whole fields (do not include velocity), and the corresponding variables without stars are reserved for their anomalies. The subscript h indicates the component on the p plane; for example, $\mathbf{v} = (\mathbf{v}_h, \omega)$, $\nabla = (\nabla_h, \partial/\partial p)$, and so forth. The other symbols are conventional (see appendix A).

Let $\overline{T}$ denote the temperature averaged over the p plane and time and T denote the departure of T^* from $\overline{T}$. Then

$$T^* = \overline{T}(p) + T(\lambda, \varphi, p; t).$$
(31)

The ideal gas law [(30)], or $\alpha^* = (R/p)T^*$, implies a linear relation between T and α, and hence, equally, we have

$$\alpha^* = \overline{\alpha}(p) + \alpha(\lambda, \varphi, p; t).$$
(32)

By hydrostaticity,

$$\Phi^* = \overline{\Phi}(p_s) - \int_{p_s}^{p} \alpha^* \, dp = \overline{\Phi}(p_s)$$

$$- \int_{p_s}^{p} \overline{\alpha} \, dp - \int_{p_s}^{p} \alpha \, dp \equiv \overline{\Phi}(p) + \Phi.$$
(33)

The heat equation in (29) may then be rewritten in terms of T:

$$\frac{\partial T}{\partial t} + \mathbf{v}_h \cdot \nabla_h T + \omega \frac{\partial T}{\partial p} + \omega \frac{\partial \overline{T}}{\partial p} - \omega \frac{\alpha^*}{c_p} = \frac{\dot{q}_{\text{net}}}{c_p}.$$

But

$$\frac{1}{\alpha^*} \frac{\partial \overline{T}}{\partial p} = -\frac{1}{g} \frac{\partial p}{\partial z} \frac{\partial \overline{T}}{\partial p} = -\frac{1}{g} \frac{\partial \overline{T}}{\partial z} = \frac{1}{g} L,$$

where $L = -\partial \overline{T}/\partial z$ is the lapse rate. Also let

$$L_d \equiv \frac{g}{c_p} \approx 9.8 \times 10^{-3} \, \text{K m}^{-1}$$

(i.e., lapse rate for dry air). The above equation hence becomes

$$\frac{\partial T}{\partial t} + \mathbf{v}_h \cdot \nabla_h T + \omega \frac{\partial T}{\partial p} + \omega \alpha^* \frac{L - L_d}{g} = \frac{\dot{q}_{\text{net}}}{c_p}.$$
(34)

Note that

$$\alpha^* \frac{L - L_d}{g} = \alpha^* \left(-\frac{\partial \overline{T}}{\partial z} - \frac{1}{c_p} \right) = \frac{R\overline{T}}{c_p p} - \frac{\partial \overline{T}}{\partial p} = -\frac{\overline{T}}{\theta} \frac{\partial \theta}{\partial p} \equiv S_p$$

is the stability parameter (θ is the potential temperature).

From above, we also have

$$\nabla_h \Phi^* = \nabla_h \Phi,$$
(35)

and by the hydrostatic assumption,

$$\frac{\partial \Phi}{\partial p} = \frac{\partial \Phi^*}{\partial p} - \frac{\partial \overline{\Phi}}{\partial p} = -\alpha^* + \overline{\alpha} = -\alpha.$$
(36)

Hence the primitive equations are, in terms of T, Φ, etc.,

$$\frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h \cdot \nabla_h \mathbf{v}_h + \omega \frac{\partial \mathbf{v}_h}{\partial p} + f\mathbf{k} \times \mathbf{v}_h = -\nabla_h \Phi + \mathbf{F}_{m,p} + \mathbf{F}_{m,h},$$
(37)

$$\frac{\partial \Phi}{\partial p} = -\alpha,$$
(38)

$$\nabla_h \cdot \mathbf{v}_h + \frac{\partial \omega}{\partial p} = 0,$$
(39)

$$\frac{\partial T}{\partial t} + \mathbf{v}_h \cdot \nabla_h T + \omega \frac{\partial T}{\partial p} + \omega \overline{\alpha} \frac{L - L_d}{g}$$
$$+ \omega \alpha \frac{L - L_d}{g} = \frac{\dot{q}_{\text{net}}}{c_p}, \quad \text{and}$$
(40)

$$\alpha = \frac{R}{p} T.$$
(41)

In the heat equation $\omega \alpha[(L - L_d)/g]$ makes a correction term and is by comparison small (since $\alpha \ll \overline{\alpha}$).

b. Multiscale kinetic energy equations

The first step is to find $\mathbf{Q}_{K,n}^{\varpi}$, the flux on scale window ϖ at step n. This has been fulfilled in the preceding section, which we rewrite here for reference,

$$\mathbf{Q}_K^{\varpi} = \frac{1}{2} \widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim \varpi} \cdot \hat{\mathbf{v}}_h^{\sim \varpi}.$$
(42)

Componentwise this is

$$Q_{K,\lambda}^{\varpi} = \frac{1}{2} [\widehat{(uu)}^{\sim \varpi} \hat{u}^{\sim \varpi} + \widehat{(uv)}^{\sim \varpi} \hat{v}^{\sim \varpi}],$$
(43)

$$Q_{K,\varphi}^{\varpi} = \frac{1}{2} [\widehat{(vu)}^{\sim \varpi} \hat{u}^{\sim \varpi} + \widehat{(vv)}^{\sim \varpi} \hat{v}^{\sim \varpi}], \quad \text{and}$$
(44)

$$Q_{K,p}^{\varpi} = \frac{1}{2} [\widehat{(\omega u)}^{\sim \varpi} \hat{u}^{\sim \varpi} + \widehat{(\omega v)}^{\sim \varpi} \hat{v}^{\sim \varpi}].$$
(45)

From the horizontal momentum equations, the canonical transfer is

$$\Gamma_K^{\varpi} = -\widehat{(\mathbf{v} \cdot \nabla \mathbf{v}_h)}^{\sim \varpi} \cdot \hat{\mathbf{v}}_h^{\sim \varpi} + \nabla \cdot \mathbf{Q}_K^{\varpi}.$$

It is better expressed, with the aid of the incompressibility equation [(39)], as

$$\Gamma_K^{\varpi} = -[\nabla \cdot \widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim \varpi}] \cdot \hat{\mathbf{v}}_h^{\sim \varpi} + \nabla \cdot \mathbf{Q}_K^{\varpi},$$
$$= -[\nabla \cdot \widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim \varpi}] \cdot \hat{\mathbf{v}}_h^{\sim \varpi} + \frac{1}{2} \nabla \cdot [\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim \varpi} \cdot \hat{\mathbf{v}}_h^{\sim \varpi}]$$
(46)

$$= \frac{1}{2} \{\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim \varpi} : \nabla \hat{\mathbf{v}}_h^{\sim \varpi} - [\nabla \cdot \widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim \varpi}] \cdot \hat{\mathbf{v}}_h^{\sim \varpi}\},$$
(47)

where the colon operator is defined such that, for two dyadic products $\mathbf{AB}$ and $\mathbf{CD}$,

$$(\mathbf{AB}):(\mathbf{CD}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}).$$

In fact, the above can be expanded in terms of the components of $\mathbf{v}_h = (u, v)$; that is,

$$\Gamma_K^{\varpi} = \frac{1}{2} \{\widehat{(\mathbf{v}u)}^{\sim \varpi} \nabla \hat{u}^{\sim \varpi} - [\nabla \cdot \widehat{(\mathbf{v}u)}^{\sim \varpi}] \hat{u}^{\sim \varpi}\}$$
$$+ \frac{1}{2} \{\widehat{(\mathbf{v}v)}^{\sim \varpi} \nabla \hat{v}^{\sim \varpi} - [\nabla \cdot \widehat{(\mathbf{v}v)}^{\sim \varpi}] \hat{v}^{\sim \varpi}\}.$$
(48)

Notice that this is just the sum of two canonical transfers and is, hence, canonical.

The equation governing the evolution of $K^{\varpi} = (1/2)\hat{\mathbf{v}}_h^{\sim \varpi} \cdot \hat{\mathbf{v}}_h^{\sim \varpi}$ is, therefore {after $\hat{\mathbf{v}}_h^{\sim \varpi} \cdot$ [MWT of (37)]},

$$\frac{\partial K^{\varpi}}{\partial t} + \nabla \cdot \mathbf{Q}_K^{\varpi} = \Gamma_K^{\varpi} - \nabla \cdot (\hat{\mathbf{v}}^{\sim \varpi} \hat{\Phi}^{\sim \varpi}) - \hat{\omega}^{\sim \varpi} \hat{\alpha}^{\sim \varpi} + F_{K,p}^{\varpi} + F_{K,h}^{\varpi}$$
$$= \Gamma_K^{\varpi} - \nabla \cdot \mathbf{Q}_P^{\varpi} - b^{\varpi} + F_{K,p}^{\varpi} + F_{K,h}^{\varpi}.$$
(49)

Here $\mathbf{Q}_P^{\varpi} = \hat{\mathbf{v}}^{\sim \varpi} \hat{\Phi}^{\sim \varpi}$, and $b^{\varpi} = \hat{\omega}^{\sim \varpi} \hat{\alpha}^{\sim \varpi}$ is the rate of buoyancy conversion.

It is necessary to derive the expressions in spherical coordinates. If the vertical coordinate is z, then the Lamé's coefficients are $h_\lambda \approx a \cos\varphi$, $h_\varphi \approx a$, and $h_z = 1$, where a is the radius of Earth and λ and φ are longitude and latitude, respectively; thus, the divergence of $\mathbf{Q}_K^{\varpi} = (Q_{K,\lambda}^{\varpi}, Q_{K,\varphi}^{\varpi}, Q_{K,z}^{\varpi})$ is

$$\nabla \cdot \mathbf{Q} = \frac{1}{h_\lambda h_\varphi h_z} \left[\frac{\partial(h_\varphi h_z Q_{K,\lambda}^{\varpi})}{\partial \lambda} + \frac{\partial(h_\lambda h_z Q_{K,\varphi}^{\varpi})}{\partial \varphi} + \frac{\partial(h_\lambda h_\varphi Q_{K,z}^{\varpi})}{\partial z} \right]$$
$$= \frac{1}{a \cos\varphi} \frac{\partial Q_{K,\lambda}^{\varpi}}{\partial \lambda} + \frac{1}{a \cos\varphi} \frac{\partial(Q_{K,\varphi}^{\varpi} \cos\varphi)}{\partial \varphi} + \frac{\partial Q_{K,z}^{\varpi}}{\partial z}.$$
(50)

If the vertical coordinate is p, $\nabla \cdot \mathbf{Q}_K^{\varpi}$ can also be approximately expressed as

$$\nabla \cdot \mathbf{Q}_K^{\varpi} = \frac{1}{a \cos\varphi} \frac{\partial Q_{K,\lambda}^{\varpi}}{\partial \lambda} + \frac{1}{a \cos\varphi} \frac{\partial(Q_{K,\varphi}^{\varpi} \cos\varphi)}{\partial \varphi} + \frac{\partial Q_{K,p}^{\varpi}}{\partial p}.$$
(51)

For the components of $\mathbf{Q}_K^\varpi$, refer to (43)–(45). Note that this is just an approximate expression, as this is not strictly an orthogonal frame. However, since the shell of the atmosphere is thin (shallow-water assumption), the p direction may be viewed as unaffected, just as in the geographic coordinate system. Likewise,

$$\nabla \cdot \mathbf{Q}_P^\varpi = \frac{1}{a \cos\varphi} \frac{\partial(\hat{u}^{\sim\varpi}\hat{\Phi}^{\sim\varpi})}{\partial\lambda}$$
$$+ \frac{1}{a \cos\varphi} \frac{\partial(\hat{v}^{\sim\varpi}\hat{\Phi}^{\sim\varpi}\cos\varphi)}{\partial\varphi} + \frac{\partial(\hat{\omega}^{\sim\varpi}\hat{\Phi}^{\sim\varpi})}{\partial p}. \quad (52)$$

The difficulty is with the transfer term. It would be easier to start from (46). By the result of appendix C,

$$\nabla \cdot (\mathbf{vv}) = \left\{ \frac{1}{a\cos\varphi}\left[\frac{\partial u^2}{\partial\lambda} - uv\sin\varphi + u\omega\cos\varphi + \frac{\partial(vu\cos\varphi)}{\partial\varphi}\right] + \frac{\partial\omega u}{\partial p}\right\}\mathbf{e}_\lambda$$
$$+ \left\{ \frac{1}{a\cos\varphi}\left[\frac{\partial uv}{\partial\lambda} + u^2\sin\varphi + \frac{\partial(v^2\cos\varphi)}{\partial\varphi} + v\omega\cos\varphi\right] + \frac{\partial\omega v}{\partial p}\right\}\mathbf{e}_\varphi$$
$$+ \left\{ \frac{1}{a\cos\varphi}\left[\frac{\partial u\omega}{\partial\lambda} - u^2\cos\varphi + \frac{\partial(v\omega\cos\varphi)}{\partial\varphi} - v^2\cos\varphi\right] + \frac{\partial\omega^2}{\partial p}\right\}\mathbf{e}_p. \quad (53)$$

In particular,

$$\nabla \cdot (\mathbf{vv}_h) = \left[\frac{1}{a\cos\varphi}\left(\frac{\partial u^2}{\partial\lambda} - uv\sin\varphi + \frac{\partial uv\cos\varphi}{\partial\varphi}\right) + \frac{\partial\omega u}{\partial p}\right]\mathbf{e}_\lambda$$
$$+ \left[\frac{1}{a\cos\varphi}\left(\frac{\partial uv}{\partial\lambda} + u^2\sin\varphi + \frac{\partial v^2\cos\varphi}{\partial\varphi}\right) + \frac{\partial\omega v}{\partial p}\right]\mathbf{e}_\varphi - \left(\frac{u^2 + v^2}{a}\right)\mathbf{e}_p.$$

So

$$\Gamma_K^\varpi = -[\nabla \cdot \widehat{(\mathbf{vv}_h)}^{\sim\varpi}] \cdot \hat{\mathbf{v}}_h^{\sim\varpi} + \nabla \cdot \mathbf{Q}_K^\varpi$$

$$= -\left\{\frac{1}{a\cos\varphi}\left[\frac{\partial\widehat{(u^2)}^{\sim\varpi}}{\partial\lambda} - \widehat{(uv)}^{\sim\varpi}\sin\varphi + \frac{\partial\widehat{(uv)}^{\sim\varpi}\cos\varphi}{\partial\varphi}\right] + \frac{\partial\widehat{(\omega u)}^{\sim\varpi}}{\partial p}\right\}\hat{u}^{\sim\varpi}$$

$$-\left\{\frac{1}{a\cos\varphi}\left[\frac{\partial\widehat{(uv)}^{\sim\varpi}}{\partial\lambda} + \widehat{(u^2)}^{\sim\varpi}\sin\varphi + \frac{\partial\widehat{(v^2)}^{\sim\varpi}\cos\varphi}{\partial\varphi}\right] + \frac{\partial\widehat{(\omega v)}^{\sim\varpi}}{\partial p}\right\}\hat{v}^{\sim\varpi}$$

$$+ \frac{1}{2}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}[\widehat{(uu)}^{\sim\varpi}\hat{u}^{\sim\varpi} + \widehat{(uv)}^{\sim\varpi}\hat{v}^{\sim\varpi}] + \frac{1}{2}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{\cos\varphi[\widehat{(vu)}^{\sim\varpi}\hat{u}^{\sim\varpi} + \widehat{(vv)}^{\sim\varpi}\hat{v}^{\sim\varpi}]\}$$

$$+ \frac{1}{2}\frac{\partial}{\partial p}[\widehat{(\omega u)}^{\sim\varpi}\hat{u}^{\sim\varpi} + \widehat{(\omega v)}^{\sim\varpi}\hat{v}^{\sim\varpi}]$$

$$= \frac{1}{2a\cos\varphi}\left[\widehat{(u^2)}^{\sim\varpi}\frac{\partial\hat{u}^{\sim\varpi}}{\partial\lambda} - \hat{u}^{\sim\varpi}\frac{\partial\widehat{(u^2)}^{\sim\varpi}}{\partial\lambda}\right] + \frac{1}{2a\cos\varphi}\left[\widehat{(uv)}^{\sim\varpi}\frac{\partial\hat{v}^{\sim\varpi}}{\partial\lambda} - \hat{v}^{\sim\varpi}\frac{\partial\widehat{(uv)}^{\sim\varpi}}{\partial\lambda}\right]$$

$$+ \frac{1}{2a\cos\varphi}\left[\widehat{(uv)}^{\sim\varpi}\cos\varphi\frac{\partial\hat{u}^{\sim\varpi}}{\partial\varphi} - \hat{u}^{\sim\varpi}\frac{\partial\widehat{(uv)}^{\sim\varpi}}{\partial\varphi}\cos\varphi\right] + \frac{1}{2a\cos\varphi}\left[\widehat{(v^2)}^{\sim\varpi}\cos\varphi\frac{\partial\hat{v}^{\sim\varpi}}{\partial\varphi} - \hat{v}^{\sim\varpi}\frac{\partial\widehat{(v^2)}^{\sim\varpi}}{\partial\varphi}\cos\varphi\right]$$

$$+ \frac{1}{2}\left[\widehat{(\omega u)}^{\sim\varpi}\frac{\partial\hat{u}^{\sim\varpi}}{\partial p} - \hat{u}^{\sim\varpi}\frac{\partial\widehat{(\omega u)}^{\sim\varpi}}{\partial p}\right] + \frac{1}{2}\left[\widehat{(\omega v)}^{\sim\varpi}\frac{\partial\hat{v}^{\sim\varpi}}{\partial p} - \hat{v}^{\sim\varpi}\frac{\partial\widehat{(\omega v)}^{\sim\varpi}}{\partial p}\right] + \frac{\tan\varphi}{a}[\hat{u}^{\sim\varpi}\widehat{(uv)}^{\sim\varpi} - \hat{v}^{\sim\varpi}\widehat{(u^2)}^{\sim\varpi}]. \quad (54)$$

Obviously, the first six brackets are all in canonical form as shown in section 3 and, hence, represent canonical transfers. For the last term, by the property of marginalization (note here the dependence on n is suppressed),

$$\sum_{\varpi}\sum_{n}[\hat{u}^{\sim\varpi}\widehat{(uv)}^{\sim\varpi} - \hat{v}^{\sim\varpi}\widehat{(u^2)}^{\sim\varpi}] = \overline{u(uv)} - \overline{v(u^2)} = 0.$$

So they, as a whole, make Γ_K^ϖ a canonical transfer. The above formula can be further reduced to

$$\Gamma_K^\varpi = \frac{1}{2a\cos\varphi}\left[\widehat{(u^2)}^{\sim\varpi}\frac{\partial\hat{u}^{\sim\varpi}}{\partial\lambda} - \hat{u}^{\sim\varpi}\frac{\partial\widehat{(u^2)}^{\sim\varpi}}{\partial\lambda} + \widehat{(uv)}^{\sim\varpi}\frac{\partial\hat{v}^{\sim\varpi}}{\partial\lambda} - \hat{v}^{\sim\varpi}\frac{\partial\widehat{(uv)}^{\sim\varpi}}{\partial\lambda}\right]$$

$$+ \frac{1}{2a}\left[\widehat{(uv)}^{\sim\varpi}\frac{\partial\hat{u}^{\sim\varpi}}{\partial\varphi} - \hat{u}^{\sim\varpi}\frac{\partial\widehat{(uv)}^{\sim\varpi}}{\partial\varphi} + \widehat{(v^2)}^{\sim\varpi}\frac{\partial\hat{v}^{\sim\varpi}}{\partial\varphi} - \hat{v}^{\sim\varpi}\frac{\partial\widehat{(v^2)}^{\sim\varpi}}{\partial\varphi}\right]$$

$$+ \frac{1}{2}\left[\widehat{(u\omega)}^{\sim\varpi}\frac{\partial\hat{u}^{\sim\varpi}}{\partial p} - \hat{u}^{\sim\varpi}\frac{\partial\widehat{(u\omega)}^{\sim\varpi}}{\partial p} + \widehat{(v\omega)}^{\sim\varpi}\frac{\partial\hat{v}^{\sim\varpi}}{\partial p} - \hat{v}^{\sim\varpi}\frac{\partial\widehat{(v\omega)}^{\sim\varpi}}{\partial p}\right]$$

$$+ \frac{3}{2a}\tan\varphi[\hat{u}^{\sim\varpi}\widehat{(uv)}^{\sim\varpi} - \hat{v}^{\sim\varpi}\widehat{(u^2)}^{\sim\varpi}] + \frac{1}{2a}\hat{v}^{\sim\varpi}\tan\varphi[\widehat{(u^2)}^{\sim\varpi} + \widehat{(v^2)}^{\sim\varpi}]. \quad (55)$$

Note that in computing Γ^ϖ, we just need to perform the MWT of nine variables—namely, the six distinct entries of the matrix

$$\begin{pmatrix} u^2 & uv & u\omega \\ uv & v^2 & v\omega \\ u\omega & v\omega & \omega^2 \end{pmatrix}$$

plus u, v, and ω. The expression of Γ^ϖ, albeit complex, is a combination of these variables. The other terms can be easily expressed.

c. Multiscale available potential energy equation

Following the tradition since Lorenz (1955), APE is defined as

$$A = \frac{1}{2}\frac{g}{\overline{T}(L_d - L)}T^2 \equiv \frac{1}{2}cT^2, \quad (56)$$

where

$$c = \frac{g}{\overline{T}(L_d - L)} = \frac{g}{\overline{T}(g/c_p - L)}. \quad (57)$$

Originally Lorenz examined the quantity in a bulk form; we relieve the integration to define a local APE. Besides, we multiply it by g to ensure a dimension consistent with that of the kinetic energy in the preceding section.

Multiply the heat equation in (40) by cT to get

$$\frac{\partial A}{\partial t} + cT\nabla\cdot(\mathbf{v}T) + T\omega\overline{\alpha}c\frac{L-L_d}{g} + T\omega\alpha c\frac{L-L_d}{g}$$

$$= cT\frac{\dot{q}_{net}}{c_p}.$$

Or

$$\frac{\partial A}{\partial t} + \nabla\cdot(\mathbf{v}A) = \alpha\omega + \frac{T}{\overline{T}}\alpha\omega + A\omega\frac{\partial\log c}{\partial p} + cT\frac{\dot{q}_{net}}{c_p}, \quad (58)$$

where $b = \alpha\omega$ is the buoyancy conversion rate, $(T/\overline{T})b \ll b$ is the correction term, and $A\omega(\partial\log c/\partial p)$ is the apparent source/sink due to the background temperature profile. In the course of derivation, the ideal gas law $\overline{\alpha}/\overline{T} = R/p$ has been used.

To arrive at the multiscale APE equation, take an MWT on both sides of (40), followed by a multiplication with $c\hat{T}^{\sim\varpi}$. This gives

$$\frac{\partial A^\varpi}{\partial t} + c\hat{T}^{\sim\varpi}\nabla\cdot\widehat{(\mathbf{v}T)}^{\sim\varpi}$$

$$= \hat{T}^{\sim\varpi}\hat{\omega}^{\sim\varpi}\frac{\overline{\alpha}}{\overline{T}} + \hat{T}^{\sim\varpi}\widehat{(\omega\alpha)}^{\sim\varpi}\frac{1}{\overline{T}} + \frac{c}{c_p}\hat{T}^{\sim\varpi}\widehat{(\dot{q}_{net})}^{\sim\varpi}.$$

Write the source term as

$$F_A^\varpi = \frac{c}{c_p}\hat{T}^{\sim\varpi}\widehat{(\dot{q}_{net})}^{\sim\varpi},$$

and let

$$b^\varpi = \frac{\overline{\alpha}}{\overline{T}}\hat{T}^{\sim\varpi}\hat{\omega}^{\sim\varpi} = \frac{R}{p}\hat{T}^{\sim\varpi}\hat{\omega}^{\sim\varpi} = \hat{\alpha}^{\sim\varpi}\hat{\omega}^{\sim\varpi} \quad \text{and} \quad (59)$$

$$SA' = \frac{1}{\overline{T}}\hat{T}^{\sim\varpi}\widehat{(\omega\alpha)}^{\sim\varpi} = \frac{R}{p}\hat{T}^{\sim\varpi}\widehat{(\omega T)}^{\sim\varpi}\frac{1}{\overline{T}}, \quad (60)$$

where b^ϖ is the buoyancy conversion rate and the other is its correction term. Further, separate the flux from the transfer terms:

$$\mathbf{Q}_A^\varpi = \frac{1}{2}c\widehat{(\mathbf{v}T)}^{\sim\varpi}\hat{T}^{\sim\varpi} \quad \text{and} \tag{61}$$

$$\Gamma_A^\varpi = -c\hat{T}^{\sim\varpi}\nabla\cdot\widehat{(\mathbf{v}T)}^{\sim\varpi} + \nabla\cdot\mathbf{Q}_A^\varpi - \frac{1}{2}\hat{T}^{\sim\varpi}\widehat{(\omega T)}^{\sim\varpi}\frac{\partial c}{\partial p}, \tag{62}$$

where

$$\frac{1}{2}\hat{T}^{\sim\varpi}\widehat{(\omega T)}^{\sim\varpi}\frac{\partial c}{\partial p} \equiv SA'' \tag{63}$$

is the apparent source/sink term due to the vertical variation of $c = c(\overline{T})$. This correction term makes Γ_A^ϖ canonical. To see it, notice that

$$
\begin{aligned}
\Gamma_A^\varpi &= -c\hat{T}^{\sim\varpi}\nabla\cdot\widehat{(\mathbf{v}T)}^{\sim\varpi} + \frac{1}{2}\nabla\cdot[c\hat{T}^{\sim\varpi}\widehat{(\mathbf{v}T)}^{\sim\varpi}] - \frac{1}{2}\hat{T}^{\sim\varpi}\widehat{(\omega T)}^{\sim\varpi}\frac{\partial c}{\partial p}\\
&= \frac{1}{2}c[\widehat{(\mathbf{v}_h T)}^{\sim\varpi}\cdot\nabla_h\hat{T}^{\sim\varpi} - \hat{T}^{\sim\varpi}\nabla_h\cdot\widehat{(\mathbf{v}_h T)}^{\sim\varpi}] + \frac{1}{2}c\hat{T}^{\sim\varpi}\frac{\partial\widehat{(\omega T)}^{\sim\varpi}}{\partial p} + \frac{1}{2}\widehat{(\omega T)}^{\sim\varpi}\frac{\partial c\hat{T}^{\sim\varpi}}{\partial p} - \frac{1}{2}\hat{T}^{\sim\varpi}\widehat{(\omega T)}^{\sim\varpi}\frac{\partial c}{\partial p}\\
&= \frac{c}{2}[\widehat{(\mathbf{v}_h T)}^{\sim\varpi}\cdot\nabla_h\hat{T}^{\sim\varpi} - \hat{T}^{\sim\varpi}\nabla_h\cdot\widehat{(\mathbf{v}_h T)}^{\sim\varpi}] + \frac{c}{2}\left[\widehat{(\omega T)}^{\sim\varpi}\frac{\partial\hat{T}^{\sim\varpi}}{\partial p} - \hat{T}^{\sim\varpi}\frac{\partial\widehat{(\omega T)}^{\sim\varpi}}{\partial p}\right]\\
&= \frac{c}{2}[\widehat{(\mathbf{v}T)}^{\sim\varpi}\cdot\nabla\hat{T}^{\sim\varpi} - \hat{T}^{\sim\varpi}\nabla\cdot\widehat{(\mathbf{v}T)}^{\sim\varpi}],
\end{aligned}
\tag{64}
$$

which is precisely in the canonical form. Following the proof in the preceding section, it is easy to show that $\sum_n\sum_\varpi\Gamma_A^\varpi = 0$.

Combine S' and S'' as one apparent source term to give

$$S_A^\varpi = SA' + SA'' = \frac{1}{2}\hat{T}^{\sim\varpi}\widehat{(\omega T)}^{\sim\varpi}\frac{\partial c}{\partial p} + \frac{1}{\overline{T}}\hat{T}^{\sim\varpi}\widehat{(\omega\alpha)}^{\sim\varpi}. \tag{65}$$

In real applications, this is usually negligible. The multiscale APE equation now becomes

$$\frac{\partial A^\varpi}{\partial t} + \nabla\cdot\mathbf{Q}_A^\varpi = \Gamma_A^\varpi + b^\varpi + S_A^\varpi + F_A^\varpi. \tag{66}$$

In the spherical coordinates,

$$\nabla\cdot\mathbf{Q}_A^\varpi = \frac{c}{2a\cos\varphi}\frac{\partial[\widehat{(uT)}^{\sim\varpi}\hat{T}^{\sim\varpi}]}{\partial\lambda} + \frac{c}{2a\cos\varphi}\frac{\partial[\widehat{(vT)}^{\sim\varpi}\hat{T}^{\sim\varpi}\cos\varphi]}{\partial\varphi} + \frac{1}{2}\frac{\partial[c\widehat{(\omega T)}^{\sim\varpi}\hat{T}^{\sim\varpi}]}{\partial p} \quad \text{and} \tag{67}$$

$$
\begin{aligned}
\Gamma_A^\varpi = \frac{c}{2}\Bigg\{ &\frac{1}{a\cos\varphi}\widehat{(uT)}^{\sim\varpi}\frac{\partial\hat{T}^{\sim\varpi}}{\partial\lambda} + \frac{1}{a}\widehat{(vT)}^{\sim\varpi}\frac{\partial\hat{T}^{\sim\varpi}}{\partial\varphi} + \widehat{(\omega T)}^{\sim\varpi}\frac{\partial\hat{T}^{\sim\varpi}}{\partial p}\\
&- \frac{1}{a\cos\varphi}\hat{T}^{\sim\varpi}\frac{\partial\widehat{(uT)}^{\sim\varpi}}{\partial\lambda} - \frac{1}{a\cos\varphi}\hat{T}^{\sim\varpi}\frac{\partial[\widehat{(vT)}^{\sim\varpi}\cos\varphi]}{\partial\varphi} - \hat{T}^{\sim\varpi}\frac{\partial\widehat{(\omega T)}^{\sim\varpi}}{\partial p}\Bigg\}.
\end{aligned}
\tag{68}
$$

d. A note on the units

Currently the energetic terms have the units of meters squared per second cubed, if the SI base units are used. However, caution should be used when total or regional subtotal energetics are to be computed. Since here density is not a constant, one cannot just integrate the local fields with respect to a volume to obtain the bulk energetics. If the system is a Cartesian one, this will be problematic, since $(1/2)\rho\mathbf{v}_h\cdot\mathbf{v}_h$ is not a quadratic variable; the variation of ρ must also be taken into account in the above derivations.

This is, however, avoidable in an isobaric frame. An integration with respect to the "volume" form $dxdy(-dp)$ yields the real energy multiplied by a constant g.

e. Wrap-up

To wrap up, the multiscale kinetic and available energy equations are

$$\frac{\partial K^\varpi}{\partial t} + \nabla\cdot\mathbf{Q}_K^\varpi = \Gamma_K^\varpi - \nabla\cdot\mathbf{Q}_P^\varpi - b^\varpi + F_{K,p}^\varpi + F_{K,h}^\varpi \tag{69}$$

and

$$\frac{\partial A^\varpi}{\partial t} + \nabla\cdot\mathbf{Q}_A^\varpi = \Gamma_A^\varpi + b^\varpi + S_A^\varpi + F_A^\varpi. \tag{70}$$

It should be mentioned that all the terms are to be multiplied by a constant factor 2^{j_2}, where j_2 is the upper bound of the scale level of the smallest scale window. For reference, the expressions for the energetics are

TABLE 1. Multiscale energetics for the atmospheric circulation ($m^2\,s^{-3}$ if SI base units are used). If total or regional total energies (W) are to be computed, the resulting integrals with respect to (x, y, p) should be divided by g. All terms are to be multiplied by 2^{j_2}.

K^ϖ	$\frac{1}{2}\hat{\mathbf{v}}_h^{\sim\varpi} \cdot \hat{\mathbf{v}}_h^{\sim\varpi}$	KE on scale window ϖ
$\mathbf{Q}_K^\varpi$	$\frac{1}{2}\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi} \cdot \hat{\mathbf{v}}_h^{\sim\varpi}$	Flux of KE on window ϖ
Γ_K^ϖ	$\frac{1}{2}[\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi} : \nabla\hat{\mathbf{v}}_h^{\sim\varpi} - \nabla \cdot \widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi} \cdot \hat{\mathbf{v}}_h^{\sim\varpi}]$	Canonical transfer of KE to window ϖ
$\mathbf{Q}_P^\varpi$	$\hat{\mathbf{v}}^{\sim\varpi}\hat{\Phi}^{\sim\varpi}$	Pressure flux
b^ϖ	$\hat{\omega}^{\sim\varpi}\hat{\alpha}^{\sim\varpi}$	Buoyancy conversion
A^ϖ	$\frac{1}{2}c(\hat{T}^{\sim\varpi})^2, c = \dfrac{g}{\overline{\overline{T}}(g/c_p - L)}$	APE on scale window ϖ
$\mathbf{Q}_A^\varpi$	$\frac{1}{2}c\hat{T}^{\sim\varpi}\widehat{(\mathbf{v}T)}^{\sim\varpi}$	Flux of APE on window ϖ
Γ_A^ϖ	$\frac{c}{2}[\widehat{(\mathbf{v}T)}^{\sim\varpi} \cdot \nabla\hat{T}^{\sim\varpi} - \hat{T}^{\sim\varpi}\nabla \cdot \widehat{(\mathbf{v}T)}^{\sim\varpi}]$	Canonical transfer of APE to window ϖ
S_A^ϖ	$\frac{1}{2}\hat{T}^{\sim\varpi}\widehat{(\omega T)}^{\sim\varpi}\dfrac{\partial c}{\partial p} + \dfrac{1}{\overline{\overline{T}}}\widehat{(\omega\alpha)}^{\sim\varpi}$	Apparent source/sink (usually negligible)

tabulated in Table 1. Also tabulated are the expressions in spherical coordinates (λ, φ, p) (Table 2).

The energy flow for a multiple-scale window decomposition is schematized in Fig. 3. As is seen, canonical transfers mediate between the scale windows; they represent the interscale processes such as instabilities. In contrast, buoyancy conversions and transports function only within the respective individual windows; the former bring together the two types of energy—namely, APE and KE—while that latter allow different spatial locations to communicate.

5. Multiscale oceanic energetics

a. Primitive equations

The multiscale ocean energy equations have been derived in LR05. We incorporate them here for completeness, together with some modification and correction.

For an incompressible and hydrostatic Boussinesq fluid flow, the primitive equations are

$$\frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h \cdot \nabla_h\mathbf{v}_h + w\frac{\partial \mathbf{v}_h}{\partial z} + f\mathbf{k} \times \mathbf{v}_h = -\frac{1}{\rho_0}\nabla_h P + \mathbf{F}_{m,z} + \mathbf{F}_{m,h},$$
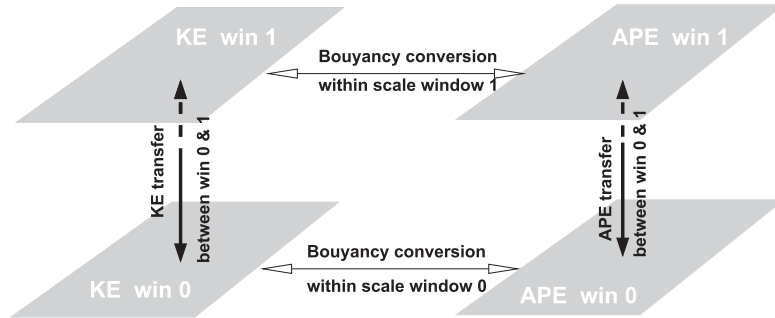
(71)

$$\frac{\partial P}{\partial z} = -\rho g,$$ (72)

$$\nabla_h \cdot \mathbf{v}_h + \frac{\partial w}{\partial z} = 0, \quad \text{and}$$ (73)

$$\frac{\partial \rho}{\partial t} + \mathbf{v}_h \cdot \nabla_h\rho + w\frac{\partial \rho}{\partial z} = \frac{N^2\rho_0}{g}w + F_{\rho,z} + F_{\rho,h},$$ (74)

TABLE 2. Expansion of the canonical transfers in Table 1 in spherical coordinates.

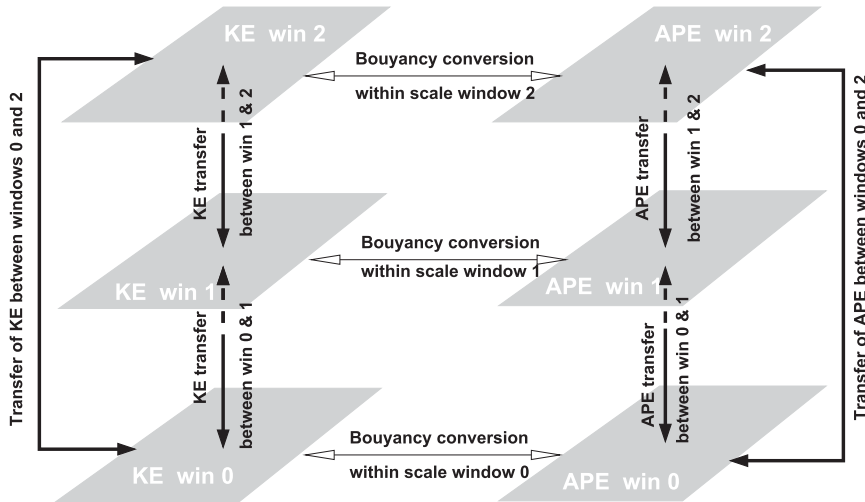|	
Γ_K^ϖ	$\dfrac{1}{2a\cos\varphi}\left[\widehat{(u^2)}^{\sim\varpi}\dfrac{\partial\hat{u}^{\sim\varpi}}{\partial\lambda} - \hat{u}^{\sim\varpi}\dfrac{\partial\widehat{(u^2)}^{\sim\varpi}}{\partial\lambda} + \widehat{(uv)}^{\sim\varpi}\dfrac{\partial\hat{v}^{\sim\varpi}}{\partial\lambda} - \hat{v}^{\sim\varpi}\dfrac{\partial\widehat{(uv)}^{\sim\varpi}}{\partial\lambda}\right]$
	$+ \dfrac{1}{2a}\left[\widehat{(uv)}^{\sim\varpi}\dfrac{\partial\hat{u}^{\sim\varpi}}{\partial\varphi} - \hat{u}^{\sim\varpi}\dfrac{\partial\widehat{(uv)}^{\sim\varpi}}{\partial\varphi} + \widehat{(v^2)}^{\sim\varpi}\dfrac{\partial\hat{v}^{\sim\varpi}}{\partial\varphi} - \hat{v}^{\sim\varpi}\dfrac{\partial\widehat{(v^2)}^{\sim\varpi}}{\partial\varphi}\right]$
	$+ \dfrac{1}{2}\left[\widehat{(u\omega)}^{\sim\varpi}\dfrac{\partial\hat{u}^{\sim\varpi}}{\partial p} - \hat{u}^{\sim\varpi}\dfrac{\partial\widehat{(u\omega)}^{\sim\varpi}}{\partial p} + \widehat{(v\omega)}^{\sim\varpi}\dfrac{\partial\hat{v}^{\sim\varpi}}{\partial p} - \hat{v}^{\sim\varpi}\dfrac{\partial\widehat{(v\omega)}^{\sim\varpi}}{\partial p}\right]$
	$+ \dfrac{3}{2a}\tan\varphi[\hat{u}^{\sim\varpi}\widehat{(uv)}^{\sim\varpi} - \hat{v}^{\sim\varpi}\widehat{(u^2)}^{\sim\varpi}]$
	$+ \dfrac{1}{2a}\hat{v}^{\sim\varpi}\tan\varphi[\widehat{(u^2)}^{\sim\varpi} + \widehat{(v^2)}^{\sim\varpi}]$
Γ_A^ϖ	$\dfrac{c}{2}\left\{\dfrac{1}{a\cos\varphi}\widehat{(uT)}^{\sim\varpi}\dfrac{\partial\hat{T}^{\sim\varpi}}{\partial\lambda} + \dfrac{1}{a}\widehat{(vT)}^{\sim\varpi}\dfrac{\partial\hat{T}^{\sim\varpi}}{\partial\varphi} + \widehat{(\omega T)}^{\sim\varpi}\dfrac{\partial\hat{T}^{\sim\varpi}}{\partial p}\right.$
	$\left. - \dfrac{1}{a\cos\varphi}\hat{T}^{\sim\varpi}\dfrac{\partial\widehat{(uT)}^{\sim\varpi}}{\partial\lambda} - \dfrac{1}{a\cos\varphi}\hat{T}^{\sim\varpi}\dfrac{\partial[\widehat{(vT)}^{\sim\varpi}\cos\varphi]}{\partial\varphi} - \hat{T}^{\sim\varpi}\dfrac{\partial\widehat{(\omega T)}^{\sim\varpi}}{\partial p}\right\}$

FIG. 3. Schematic of the energy flow for (a) two-window and (b) three-window decomposition. Clearly, buoyancy conversions take place within their respective scale windows; they are not indicators of instabilities. For clarity, the transfers $\Gamma^{0\oplus2\to1}$, $\Gamma^{1\oplus2\to0}$, and $\Gamma^{0\oplus1\to2}$ are not drawn in (b). The transports of APE, KE, and pressure also take place within their respective scale windows only (not shown).

where the subgrid process parameterization are symbolically written as $\mathbf{F}_m$ and F_ρ. For the other notations, refer to appendix A.

b. Multiscale APE equation

Following Lorenz's convention, available potential energy is defined to be

$$A = \frac{1}{2}\frac{g^2}{\rho_0^2 N^2}\rho^2 \equiv \frac{1}{2}c\rho^2, \tag{75}$$

where

$$c = \frac{g^2}{\rho_0^2 N^2} = \frac{g}{\rho_0 s} \quad \text{and} \quad s = -\frac{\partial\overline{\rho}(z)}{\partial z}. \tag{76}$$

For a recent careful discussion on Boussinesq approximation and potential energy, refer to Ingersoll (2005). As argued before, the multiscale APE on window ϖ at step n is $c(\hat{\rho}^{\sim\varpi})^2/2$. Take MWT on both sides of the equation of density anomaly and multiply with $c\hat{\rho}^{\sim\varpi}$. It has been shown by LR05 that, to a good approximation,

$c\hat{\rho}^{\sim\varpi}(\widehat{\partial\rho/\partial t})^{\sim\varpi}$ can be identified as $\partial A_n^\varpi/\partial t$. The resulting APE equation is, therefore,

$$\frac{\partial A_n^\varpi}{\partial t} + c\hat{\rho}^{\sim\varpi}\widehat{(\mathbf{v}\cdot\nabla_h\rho)}^{\sim\varpi} + c\hat{\rho}^{\sim\varpi}\left(\widehat{w\frac{\partial\rho}{\partial z}}\right)^{\sim\varpi}$$

$$= \frac{g}{\rho_0}\hat{\rho}^{\sim\varpi}\hat{w}^{\sim\varpi} + F_{A,z}^\varpi + F_{A_h}^\varpi,$$

where

$$\nabla_h\rho = \frac{1}{a\cos\varphi}\frac{\partial\rho}{\partial\lambda}\mathbf{e}_\lambda + \frac{1}{a}\frac{\partial\rho}{\partial\varphi}\mathbf{e}_\varphi \tag{77}$$

and

$$\frac{g}{\rho_0}\hat{\rho}^{\sim\varpi}\hat{w}^{\sim\varpi} \equiv b_n^\varpi \tag{78}$$

is the rate of buoyancy conversion.

$$\sum_\varpi\sum_n[-c\hat{\rho}^{\sim\varpi}\widehat{(\mathbf{v}\cdot\nabla\rho)}^{\sim\varpi} + \nabla\cdot\mathbf{Q}_A^\varpi] = -c\overline{\rho\nabla\cdot(\rho\mathbf{v})} + \frac{1}{2}\nabla\cdot[c\overline{\rho(\mathbf{v}\rho)}] = \frac{1}{2}\frac{\partial c}{\partial z}\overline{\rho^2 w},$$

where the overbar denotes averaging over the time period. To fix the problem, write

$$S_{A,n}^\varpi = \frac{1}{2}\frac{\partial c}{\partial z}\hat{\rho}^{\sim\varpi}\widehat{(\rho w)}^{\sim\varpi}, \tag{80}$$

which is the apparent source/sink due to the vertical stratification. Then

$$\Gamma_A^\varpi = [-c\hat{\rho}^{\sim\varpi}\widehat{(\mathbf{v}\cdot\nabla\rho)}^{\sim\varpi} + \nabla\cdot\mathbf{Q}_{A,n}^\varpi] - S_{A,n}^\varpi$$

$$= \frac{c}{2}[\widehat{(\mathbf{v}\rho)}^{\sim\varpi}\cdot\nabla\hat{\rho}^{\sim\varpi} - \hat{\rho}^{\sim\varpi}\nabla\cdot\widehat{(\mathbf{v}\rho)}^{\sim\varpi}] \tag{81}$$

proves to be canonical. The multiscale APE equation is, accordingly,

$$\frac{\partial A^\varpi}{\partial t} + \nabla\cdot\mathbf{Q}_A^\varpi = \Gamma_A^\varpi + b^\varpi + S_A^\varpi + F_{A,z}^\varpi + F_{A,h}^\varpi. \tag{82}$$

In the spherical coordinate frame, by (50), we have

$$\nabla\cdot\mathbf{Q}_A^\varpi = \frac{c}{2a\cos\varphi}\left\{\frac{\partial}{\partial\lambda}[\hat{\rho}^{\sim\varpi}\widehat{(u\rho)}^{\sim\varpi}] + \frac{\partial}{\partial\varphi}[\hat{\rho}^{\sim\varpi}\widehat{(v\rho)}^{\sim\varpi}\cos\varphi]\right\}$$

$$+ \frac{1}{2}\frac{\partial}{\partial z}[c\hat{\rho}^{\sim\varpi}\widehat{(w\rho)}^{\sim\varpi}]$$

$$\tag{83}$$

and

The key to the multiscale energetics formalism is the separation of flux and transfer processes. From the result of section 3a, the flux of APE by $\mathbf{v}$ at step n on window ϖ is

$$\mathbf{Q}_A^\varpi = \frac{1}{2}c\hat{\rho}^{\sim\varpi}\widehat{(\mathbf{v}\rho)}^{\sim\varpi}. \tag{79}$$

Hence the above equation can be written as

$$\frac{\partial A^\varpi}{\partial t} + \nabla\cdot\mathbf{Q}_A^\varpi = [-c\hat{\rho}^{\sim\varpi}\widehat{(\mathbf{v}\cdot\nabla\rho)}^{\sim\varpi} + \nabla\cdot\mathbf{Q}_A^\varpi]$$

$$+ b_n^\varpi + F_{A,z}^\varpi + F_{A,h}^\varpi.$$

But the bracket on the rhs is still not the canonical transfer that we are seeking for. Since $c = c(z)$, it does not summarize to zero over n and ϖ. In fact,

$$\Gamma_A^\varpi = \frac{c}{2}\left\{\frac{1}{a\cos\varphi}\widehat{(u\rho)}^{\sim\varpi}\frac{\partial\hat{\rho}^{\sim\varpi}}{\partial\lambda} + \frac{1}{a}\widehat{(v\rho)}^{\sim\varpi}\frac{\partial\hat{\rho}^{\sim\varpi}}{\partial\varphi}\right.$$

$$+ \widehat{(w\rho)}^{\sim\varpi}\frac{\partial\hat{\rho}^{\sim\varpi}}{\partial z} - \hat{\rho}^{\sim\varpi}\left[\frac{1}{a\cos\varphi}\frac{\partial\widehat{(u\rho)}^{\sim\varpi}}{\partial\lambda}\right.$$

$$\left.+ \frac{1}{a\cos\varphi}\frac{\partial\widehat{(v\rho)}^{\sim\varpi}\cos\varphi}{\partial\varphi} + \frac{\partial\widehat{(w\rho)}^{\sim\varpi}}{\partial z}\right]\right\}. \tag{84}$$

c. Multiscale KE equation

The equation governing the evolution of the multiscale KE

$$K^\varpi = \frac{1}{2}\hat{\mathbf{v}}^{\sim\varpi}\cdot\hat{\mathbf{v}}^{\sim\varpi} \tag{85}$$

can be obtained by taking MWT on both sides of the horizontal momentum equations, followed by a dot product with $\hat{\mathbf{v}}^{\sim\varpi}$. This results in

$$\frac{\partial K_n^\varpi}{\partial t} + \widehat{(\mathbf{v}\cdot\nabla\mathbf{v}_h)}^{\sim\varpi}\cdot\hat{\mathbf{v}}_h^{\sim\varpi} = -\frac{1}{\rho_0}\mathbf{v}_h\cdot\nabla_h\hat{P}^{\sim\varpi} + F_{K,z}^\varpi + F_{K,h}^\varpi$$

$$= -\mathbf{v}\cdot\nabla\frac{\hat{P}^{\sim\varpi}}{\rho_0} + \hat{w}^{\sim\varpi}\frac{\partial\hat{P}^{\sim\varpi}/\rho_0}{\partial z} + F_{K,z}^\varpi + F_{K,h}^\varpi$$

$$= -\nabla\cdot\left(\hat{\mathbf{v}}^{\sim\varpi}\frac{\hat{P}^{\sim\varpi}}{\rho_0}\right) - \frac{g}{\rho_0}\hat{\rho}^{\sim\varpi}\hat{w}^{\sim\varpi} + F_{K,z}^\varpi + F_{K,h}^\varpi$$

$$\equiv -\nabla\cdot\mathbf{Q}_P^\varpi - b^\varpi + F_{K,z}^\varpi + F_{K,h}^\varpi,$$

TABLE 3. Multiscale energetics for oceanic circulations (m^2 s^{-3} if SI base units are used). The expressions in spherical coordinates are the same in form as that in Table 2, except that the coordinate p should be replaced by z and T by ρ. All terms are to be multiplied by 2^{j_2}.

K^ϖ	$\frac{1}{2}\hat{\mathbf{v}}_h^{\sim\varpi}\cdot\hat{\mathbf{v}}_h^{\sim\varpi}$	KE on scale window ϖ
$\mathbf{Q}_K^\varpi$	$\frac{1}{2}[\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi}\cdot\hat{\mathbf{v}}_h^{\sim\varpi}]$	Flux of KE on window ϖ
Γ_K^ϖ	$\frac{1}{2}[\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi}:\nabla\hat{\mathbf{v}}_h^{\sim\varpi}-\nabla\cdot\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi}\cdot\hat{\mathbf{v}}_h^{\sim\varpi}]$	Canonical transfer of KE to window ϖ
$\mathbf{Q}_P^\varpi$	$\frac{1}{\rho_0}\hat{\mathbf{v}}^{\sim\varpi}\hat{P}^{\sim\varpi}$	Pressure flux on window ϖ
b^ϖ	$\frac{g}{\rho_0}\hat{\rho}^{\sim\varpi}\hat{w}^{\sim\varpi}$	Buoyancy conversion on window ϖ
A^ϖ	$\frac{1}{2}c(\hat{\rho}^{\sim\varpi})^2,\ c=\frac{g^2}{\rho_0^2 N^2}$	APE on window ϖ
$\mathbf{Q}_A^\varpi$	$\frac{1}{2}[c\hat{\rho}^{\sim\varpi}\widehat{(\mathbf{v}\rho)}^{\sim\varpi}]$	Flux of APE on window ϖ
Γ_A^ϖ	$\frac{c}{2}[\widehat{(\mathbf{v}\rho)}^{\sim\varpi}\cdot\nabla\hat{\rho}^{\sim\varpi}-\hat{\rho}^{\sim\varpi}\nabla\cdot\widehat{(\mathbf{v}\rho)}^{\sim\varpi}]$	Canonical transfer of APE to window ϖ
S_A^ϖ	$\frac{1}{2}\hat{\rho}^{\sim\varpi}\widehat{(\omega\rho)}^{\sim\varpi}\frac{\partial c}{\partial z}$	Apparent source/sink of A^ϖ (usually negligible)

where b^ϖ is the buoyancy conversion rate and $-\nabla\cdot\mathbf{Q}_P^\varpi$ the pressure working rate. In the above derivation, (72) and (73) (incompressibility and hydrostaticity, respectively) have been used.

By the transport–transfer separation, the above multiscale KE equation can be written as

$$\frac{\partial K^\varpi}{\partial t}+\nabla\cdot\mathbf{Q}_K^\varpi=\Gamma_K^\varpi-\nabla\cdot\mathbf{Q}_P^\varpi-b^\varpi+F_{K,z}^\varpi+F_{K,h}^\varpi, \quad (86)$$

where

$$\mathbf{Q}_K^\varpi=\frac{1}{2}\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi}\cdot\hat{\mathbf{v}}_h^{\sim\varpi} \quad (87)$$

and

$$\begin{aligned}\Gamma_K^\varpi&=-\widehat{(\mathbf{v}\cdot\nabla\mathbf{v}_h)}^{\sim\varpi}\cdot\hat{\mathbf{v}}_h^{\sim\varpi}+\nabla\cdot\mathbf{Q}_K^\varpi\\&=-\nabla\cdot\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi}\cdot\hat{\mathbf{v}}_h^{\sim\varpi}+\nabla\cdot\mathbf{Q}_{K,n}^\varpi\\&=\frac{1}{2}\{\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi}:\nabla\hat{\mathbf{v}}_h^{\sim\varpi}-[\nabla\cdot\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi}]\cdot\hat{\mathbf{v}}_h^{\sim\varpi}\},\end{aligned} \quad (88)$$

which are precisely the same as that for the atmosphere case. In the spherical coordinates, (λ,φ,z), $\nabla\cdot\mathbf{Q}_K^\varpi$, and Γ_K^ϖ are also like that in (51) and (55), except that ω should be replaced by w and p by z.

d. Wrap-up

To wrap up, the multiscale ocean energetic equations are

$$\frac{\partial A^\varpi}{\partial t}+\nabla\cdot Q_A^\varpi=\Gamma_A^\varpi+b^\varpi+S_A^\varpi+F_{A,z}^\varpi+F_{A,h}^\varpi \quad (89)$$

and

$$\frac{\partial K^\varpi}{\partial t}+\nabla\cdot Q_K^\varpi=\Gamma_{K^\varpi}-\nabla\cdot\mathbf{Q}_P^\varpi-b^\varpi+F_{K,z}^\varpi+F_{K,h}^\varpi. \quad (90)$$

Refer to Table 3 for the expressions.

6. Multiscale quasigeostrophic energetics

The multiscale energy equations like (49) and (66) cannot be directly derived from the quasigeostrophic (QG) equation. We have to go back to where the QG equation comes from and do the derivation, and this is what Pinardi and Robinson (1986) did with their regional QG energetics.

Since the atmosphere and ocean share the same QG equation, it suffices to start off the derivation from either (37)–(41) or (71)–(74). As a z coordinate is desired, we choose the latter. To simplify the presentation, the dissipative and diffusive processes are omitted. They are not essential to the derivation, and their effect may be added symbolically after the other terms are finalized. From appendix C, the QG equation we will be dealing with is

$$\frac{\partial}{\partial t}\left[\nabla_h^2\psi+\frac{\partial}{\partial z}\left(\frac{F_r^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right]+\alpha_\ell J\left\{\psi,\left[\nabla_h^2\psi+\frac{\partial}{\partial z}\left(\frac{F_r^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right]\right\}$$
$$+\beta\frac{\partial\psi}{\partial x}=0, \quad (91)$$

where F_r is the rotational internal Froude number, α_ℓ is a dimensionless measure of the importance of advection, and J is the Jacobian operator; the other notations are conventional and the reader is referred to appendix A.

a. QG kinetic energetics

The inviscid version of the KE equation [(89)] is rewritten as

$$\frac{\partial K^{\varpi}}{\partial t} + \nabla_h \cdot \mathbf{Q}_{K,h}^{\varpi} + \frac{\partial Q_{K,z}^{\varpi}}{\partial z} = \Gamma_{K,h}^{\varpi} + \Gamma_{K,z}^{\varpi}$$
$$- \nabla_h \cdot \mathbf{Q}_{P,h}^{\varpi} - \frac{\partial Q_{P,z}^{\varpi}}{\partial z} - b^{\varpi}, \quad (92)$$

where

$$\Gamma_{K,h}^{\varpi} = -\nabla_h \cdot \widehat{(\mathbf{v}_h \mathbf{v}_h)}^{\sim \varpi} \cdot \hat{\mathbf{v}}_h^{\sim \varpi} + \nabla_h \cdot \mathbf{Q}_{K,h}^{\varpi} \quad \text{and} \quad (93)$$

$$\Gamma_{K,z}^{\varpi} = -\frac{\partial \widehat{(w \mathbf{v}_h)}^{\sim \varpi}}{\partial z} \cdot \hat{\mathbf{v}}_h^{\sim \varpi} + \frac{\partial Q_{K,z}^{\varpi}}{\partial z}. \quad (94)$$

In the transport and transfer terms, the effects due to horizontal advection and vertical advection are distinguished. As we will see soon, this will greatly help simplify the QG energetics.

Using the usual scaling (e.g., McWilliams 2006),

$$(x, y) \sim L_0, \quad z \sim H_0, \quad t \sim t_0,$$
$$(u, v) \sim U_0, \quad w \sim \frac{H_0}{L_0} U_0,$$
$$f \sim f_0, \quad N \sim N_0,$$
$$P \sim U_0 f_0 \rho_0 L_0, \quad \text{and} \quad \rho \sim \frac{f_0 U_0 L_0}{g H_0} \rho_0,$$

and noticing that the multiscale window transform does not affect the scaling, it is easy to have

$$\frac{\partial K^{\varpi}}{\partial t} \sim \frac{U_0^2}{t_0};$$

$$\nabla_h \mathbf{Q}_{K,h}^{\varpi} \quad \text{and} \quad \frac{\partial Q_{K,z}^{\varpi}}{\partial z} \sim \frac{U_0^3}{L_0};$$

$$\Gamma_{K,h}^{\varpi} \quad \text{and} \quad \Gamma_{K,z}^{\varpi} \sim \frac{U_0^3}{L_0};$$

$$\nabla_h \cdot \mathbf{Q}_{P,h}^{\varpi} \quad \text{and} \quad \frac{\partial Q_{P,z}^{\varpi}}{\partial z} \sim U_0^2 f_0; \quad \text{and}$$

$$b^{\varpi} = \frac{g}{\rho_0} \hat{\rho}^{\sim \varpi} \hat{w}^{\sim \varpi} \sim U_0^2 f_0.$$

This will yield the nondimensionalized kinetic energetics.

For clarity, hereafter throughout this subsection, all variables are understood as nondimensional. From above, (92) is now reduced to its nondimensional form:

$$\varepsilon \frac{\partial K^{\varpi}}{\partial t} + \varepsilon \alpha_\ell \nabla_h \cdot \mathbf{Q}_{K,h}^{\varpi} + \varepsilon \alpha_\ell \frac{\partial Q_{K,h}^{\varpi}}{\partial z}$$
$$= \varepsilon \alpha_\ell \Gamma_{K,h}^{\varpi} + \varepsilon \alpha_\ell \Gamma_{K,z}^{\varpi} - \nabla_h \cdot \mathbf{Q}_{P,h}^{\varpi} - \frac{\partial Q_{P,z}^{\varpi}}{\partial z} - b^{\varpi}, \quad (95)$$

where $\varepsilon = 1/f_0 t_0$ is the Rossby number and $\alpha_\ell = U_0 t_0/L_0$ measures the relative importance of advection to local change. In many textbooks, α_ℓ is taken to be one, so that $\varepsilon \alpha_\ell = U_0/f_0 L_0$ is defined as the Rossby number.

As usual, expand the variables in the power of ε,

$$P = [P]_0 + \varepsilon[P]_1 + \varepsilon^2[P]_2 + \dots, \quad (96)$$
$$w = [w]_0 + \varepsilon[w]_1 + \varepsilon^2[w]_2 + \dots, \quad (97)$$
$$\mathbf{v}_h = [\mathbf{v}_h]_0 + \varepsilon[\mathbf{v}_h]_1 + \varepsilon^2[\mathbf{v}_h]_2 + \dots, \quad \text{and} \quad (98)$$
$$\rho = [\rho]_0 + \varepsilon[\rho]_1 + \varepsilon^2[\rho]_2 + \dots. \quad (99)$$

Based on these expansions, the multiscale energetic terms can also be expanded. For example,

$$K = [K]_0 + \varepsilon[K]_1 + \dots,$$

where $[K]_0 = (1/2)[\mathbf{v}_h]_0 \cdot [\mathbf{v}_h]_0$, $[K]_1 = [\mathbf{v}_h]_0 \cdot [\mathbf{v}_h]_1$, and so forth. By the classical result [see (C5)–(C7) in appendix C], $[w]_0 = 0$. So

$$\varepsilon \frac{\partial Q_{K,z}^{\varpi}}{\partial z} = \varepsilon \frac{\partial}{\partial z} \frac{1}{2} [\widehat{(w \mathbf{v}_h)}^{\sim \varpi} \cdot \mathbf{v}_h] \sim O(\varepsilon^2).$$

Likewise,

$$\varepsilon \Gamma_{K,z}^{\varpi} \sim O(\varepsilon^2),$$
$$\frac{\partial}{\partial z} Q_{P,z}^{\varpi} = \frac{\partial}{\partial z} (\hat{w}^{\sim \varpi} \hat{P}^{\sim \varpi}) = \varepsilon \frac{\partial}{\partial z} ([\hat{w}^{\sim \varpi}]_1 [\hat{P}^{\sim \varpi}]_0) + O(\varepsilon^2),$$
$$b^{\varpi} = \hat{\rho}^{\sim \varpi} \hat{w}^{\sim \varpi} = \varepsilon [\hat{\rho}^{\sim \varpi}]_0 [\hat{w}^{\sim \varpi}]_1 + O(\varepsilon^2).$$

Substituting the power expansions into (95), taking into account the above facts, and equating the terms of like power, we have, to $O(\varepsilon^0)$,

$$\nabla_h \cdot ([\hat{\mathbf{v}}_h^{\sim \varpi}]_0 [\hat{P}^{\sim \varpi}]_0) = 0.$$

So a huge part of the pressure working rate is actually zero. To $O(\varepsilon^1)$,

$$\frac{\partial [K^{\varpi}]_0}{\partial t} + \alpha_\ell \nabla_h \cdot [\mathbf{Q}_{K,h}^{\varpi}]_0 = \alpha_\ell [\Gamma_{K,h}^{\varpi}]_0 - \nabla_h \cdot [\mathbf{Q}_{P,h}^{\varpi}]_1 - \frac{\partial}{\partial z} [Q_{P,z}^{\varpi}]_1 - [b^{\varpi}]_1$$
$$= \alpha_\ell \Gamma_{K,h}^{\varpi} - \nabla_h \cdot ([\hat{\mathbf{v}}_h^{\sim \varpi}]_0 [\hat{P}^{\sim \varpi}]_1 + [\hat{\mathbf{v}}_h^{\sim \varpi}]_1 [\hat{P}^{\sim \varpi}]_0) - \frac{\partial}{\partial z} ([\hat{w}^{\sim \varpi}]_1 [\hat{P}^{\sim \varpi}]_0) - [\hat{\rho}^{\sim \varpi}]_0 [\hat{w}^{\sim \varpi}]_1.$$

To this order, $[\mathbf{v}_h]_0$ is the geostrophic flow: $[\mathbf{v}_h]_0 = \mathbf{k} \times \nabla[P]_0$ and $[\rho]_0$ is $-\partial[P]_0/\partial z$ by hydrostaticity. $[w]_1$ and $[\mathbf{v}_h]_1$ can also be obtained [see (C8)–(C9)]:

$$[w]_1 = -\frac{F_r^2}{N^2}\mathscr{L}\left(\frac{\partial[P]_0}{\partial z}\right) \quad \text{and}$$

$$[\mathbf{v}_h]_1 = \mathbf{k}\times\mathscr{L}([\mathbf{v}_h]_0) - \beta y[\mathbf{v}_h]_0 + \mathbf{k}\times\nabla_h[P]_1,$$

where $F_r = f_0L_0/N_0H_0$ is the rotational internal Froude number and $\mathscr{L}$ stands for the operator

$$\mathscr{L} \equiv \frac{\partial}{\partial t} + \alpha_\ell[\mathbf{v}_h]_0\cdot\nabla_h \qquad (100)$$

(i.e., advection by the geostrophic flow). With these, it is straightforward to compute $[K^\varpi]_0$, $[\mathbf{Q}^\varpi_{K,h}]_0$, $[\Gamma^\varpi_{K,h}]_0$, $\partial/\partial z[Q^\varpi_{P,z}]_1$, and $[b^\varpi]_1$. The difficulty comes from the horizontal pressure working rate $[\mathbf{Q}^\varpi_{P,h}]_1$, where $[\hat{P}^{\sim\varpi}]_1$ is involved. But

$$\nabla_h\cdot([\hat{\mathbf{v}}_h^{\sim\varpi}]_0[\hat{P}^{\sim\varpi}]_1 + [\hat{\mathbf{v}}_h^{\sim\varpi}]_1[\hat{P}^{\sim\varpi}]_0)$$
$$= \nabla_h\cdot([\hat{\mathbf{v}}_h^{\sim\varpi}]_0[\hat{P}^{\sim\varpi}]_1 + \{\mathbf{k}\times(\widehat{\mathscr{L}[\mathbf{v}_h]_0})^{\sim\varpi} - \beta y[\hat{\mathbf{v}}_h^{\sim\varpi}]_0 + \mathbf{k}\times\nabla_h[\hat{P}^{\sim\varpi}]_1\}[\hat{P}^{\sim\varpi}]_0)$$
$$= \nabla_h\cdot(\{\mathbf{k}\times(\widehat{\mathscr{L}[\mathbf{v}_h]_0})^{\sim\varpi} - \beta y[\hat{\mathbf{v}}_h^{\sim\varpi}]_0\}[\hat{P}^{\sim\varpi}]_0) + \nabla_h\cdot([\hat{\mathbf{v}}_h^{\sim\varpi}]_0[\hat{P}^{\sim\varpi}]_1 + \mathbf{k}\times\nabla_h[\hat{P}^{\sim\varpi}]_1[\hat{P}^{\sim\varpi}]_0).$$

Notice that the second divergence vanishes. In fact, it is

$$\nabla_h\cdot(\mathbf{k}\times\nabla_h[\hat{P}^{\sim\varpi}]_0[\hat{P}^{\sim\varpi}]_1 + \mathbf{k}\times\nabla_h[\hat{P}^{\sim\varpi}]_1[\hat{P}^{\sim\varpi}]_0)$$
$$= -\mathbf{k}\cdot\nabla_h\times(\nabla_h[\hat{P}^{\sim\varpi}]_0[\hat{P}^{\sim\varpi}]_1 + \nabla_h[\hat{P}^{\sim\varpi}]_1[\hat{P}^{\sim\varpi}]_0) = 0.$$

Hence the whole pressure working rate

$$\nabla_h\cdot[\mathbf{Q}^\varpi_{P,h}]_1$$
$$= \nabla_h\cdot(\{\mathbf{k}\times(\widehat{\mathscr{L}[\mathbf{v}_h]_0})^{\sim\varpi} - \beta y[\hat{\mathbf{v}}_h^{\sim\varpi}]_0\}[\hat{P}^{\sim\varpi}]_0). \qquad (101)$$

As a convention, denote $[P]_0$ as ψ, and for convenience, write $[\mathbf{v}_h]_0 = \mathbf{k}\times\nabla_h\psi$ as $\mathbf{v}_g$ (geostrophic velocity). Distinguishing the QG energetics terms with a subscript g, the multiscale KE now becomes

$$\frac{\partial}{\partial t}K_g^\varpi + \nabla_h\cdot\mathbf{Q}^\varpi_{g,K} = \Gamma^\varpi_{g,K} - \nabla_h\cdot\mathbf{Q}^\varpi_{g,P,h} - \frac{\partial}{\partial z}Q^\varpi_{g,P,z} - b_g^\varpi, \qquad (102)$$

where

$$K_g^\varpi = \frac{1}{2}\hat{\mathbf{v}}_g^{\sim\varpi}\cdot\hat{\mathbf{v}}_g^{\sim\varpi}, \qquad (103)$$

$$Q^\varpi_{g,K} = \alpha_\ell[Q^\varpi_{K,h}]_0 = \frac{1}{2}\alpha_\ell\widehat{(\mathbf{v}_g\mathbf{v}_g)}^{\sim\varpi}\cdot\hat{\mathbf{v}}_g^{\sim\varpi}, \qquad (104)$$

$$\Gamma^\varpi_{g,K} = \alpha_\ell[\Gamma^\varpi_{K,h}]_0$$
$$= \frac{\alpha_\ell}{2}[\widehat{(\mathbf{v}_g\mathbf{v}_g)}^{\sim\varpi}:\nabla_h\hat{\mathbf{v}}_g^{\sim\varpi} - \nabla_h\cdot\widehat{(\mathbf{v}_g\mathbf{v}_g)}^{\sim\varpi}\cdot\hat{\mathbf{v}}_g^{\sim\varpi}], \qquad (105)$$

$$\mathbf{Q}^\varpi_{g,P,h} = \left\{\mathbf{k}\times\mathscr{L}(\hat{\mathbf{v}}_g^{\sim\varpi}) - \beta y\hat{\mathbf{v}}_g^{\sim\varpi}\right\}\hat{\psi}^{\sim\varpi}, \qquad (106)$$

$$Q^\varpi_{g,P,i} = -\frac{F_r^2}{N^2}\hat{\psi}^{\sim\varpi}\widehat{\left\{\mathscr{L}\left(\frac{\partial\psi}{\partial z}\right)\right\}}^{\sim\varpi}, \quad \text{and} \qquad (107)$$

$$b_g^\varpi = \frac{F_r^2}{N^2}\frac{\partial\hat{\psi}^{\sim\varpi}}{\partial z}\widehat{\left\{\mathscr{L}\left(\frac{\partial\psi}{\partial z}\right)\right\}}^{\sim\varpi} \qquad (108)$$

(recall that all are to be multiplied by a constant fact 2^{j_2}).

b. QG available potential energetics

Rewrite the nondiffusive version of the APE equation [(90)] as

$$\frac{\partial A^\varpi}{\partial t} + \nabla_h\cdot\mathbf{Q}^\varpi_{A,h} + \frac{\partial Q^\varpi_{A,z}}{\partial z} = \Gamma^\varpi_{A,h} + \Gamma^\varpi_{A,z} + b^\varpi + S_A^\varpi. \quad (109)$$

Using the scaling as shown in the preceding subsection, we have

$$\frac{\partial A^\varpi}{\partial t} = \frac{1}{2}\frac{g^2}{\rho_0^2N^2}(\hat{\rho}^{\sim\varpi})^2 \sim \frac{1}{t_0}\frac{g^2}{\rho_0^2N_0^2}\left(\frac{f_0U_0L_0}{gH_0}\rho_0\right)^2$$
$$= \frac{U_0^2}{t_0}\frac{f_0^2L_0^2}{N_0^2H_0^2} \equiv \frac{U_0^2}{t_0}F_r^2 \quad \text{and}$$

$$\nabla_h\cdot\mathbf{Q}^\varpi_{A,h} \sim \frac{1}{L_0}\frac{g^2}{\rho_0^2N_0^2}U_0\left(\frac{f_0U_0L_0}{gH_0}\rho_0\right)^2 = \frac{U_0^3}{L_0}F_r^2,$$

where $F_r = f_0L_0/N_0H_0$ is the rotational internal Froude number (compared to the Froude number U_0/N_0H_0). Likewise, all the remaining terms, except $b^\varpi \sim U_0^2f_0$ (as given in the preceding subsection), are on the order of $(U_0^3/L_0)F_r^2$.

As in the preceding subsection, let $\varepsilon = 1/f_0t_0$ be the Rossby number and let $\alpha_\ell = U_0t_0/L_0$. The scaled nondiffusive APE equation (throughout the remainder of this subsection, all the variables are nondimensional) is, therefore,

$$\varepsilon F_r^2\left(\frac{\partial A^\varpi}{\partial t} + \alpha_\ell\nabla_h\cdot\mathbf{Q}^\varpi_{A,h} + \alpha_\ell\frac{\partial}{\partial z}Q^\varpi_{A,z}\right)$$
$$= \varepsilon F_r^2\alpha_\ell(\Gamma^\varpi_{A,h} + \Gamma^\varpi_{A,z}) + b^\varpi + \varepsilon F_r^2 S_A^\varpi. \qquad (110)$$

Usually F_r is taken as $O(1)$, but ε is small. Expanding in the power of ε, since $[w]_0 = 0$ (cf. appendix C), it is easy to show that

$$\varepsilon\frac{\partial}{\partial z}Q^\varpi_{A,z} \sim \varepsilon\Gamma^\varpi_{A,z} \sim \varepsilon S_A^\varpi \sim O(\varepsilon^2).$$

TABLE 4. Expansion of the QG energetics for (91); $\mathbf{v}_g = \mathbf{k} \times \nabla_h \psi$, $\mathscr{L} = \partial/\partial t + J(\psi, \cdot)$.

K_g^ϖ	$\dfrac{1}{2}\hat{\mathbf{v}}_g^{\sim\varpi} \cdot \hat{\mathbf{v}}_g^{\sim\varpi}$	QG KE on scale window ϖ
$Q_{g,K}^\varpi$	$\dfrac{1}{2}\alpha_\ell \widehat{(\mathbf{v}_g\mathbf{v}_g)}^{\sim\varpi} \cdot \hat{\mathbf{v}}_g^{\sim\varpi}$	Flux of QG KE within scale window ϖ
$\Gamma_{g,K}^\varpi$	$\dfrac{\alpha_\ell}{2}[\widehat{(\mathbf{v}_g\mathbf{v}_g)}^{\sim\varpi}:\nabla_h\hat{\mathbf{v}}_g^{\sim\varpi} - \nabla_h \cdot \widehat{(\mathbf{v}_g\mathbf{v}_g)}^{\sim\varpi} \cdot \hat{\mathbf{v}}_g^{\sim\varpi}]$	Canonical transfer of QG KE to window ϖ
$\mathbf{Q}_{g,P,h}^\varpi$	$\{\mathbf{k}\times[\widehat{\mathscr{L}(\mathbf{v}_g)}]^{\sim\varpi} - \beta y\hat{\mathbf{v}}_g^{\sim\varpi}\}\hat{\psi}^{\sim\varpi}$	Horizontal pressure flux on window ϖ
$Q_{g,P,z}^\varpi$	$-\dfrac{F_r^2}{N^2}\hat{\psi}^{\sim\varpi}\left[\dfrac{\partial^2\hat{\psi}^{\sim\varpi}}{\partial t\partial z} + \alpha_\ell \left(\widehat{\mathbf{v}_g \cdot \nabla_h \dfrac{\partial\psi}{\partial z}}\right)^{\sim\varpi}\right]$	Vertical pressure flux on window ϖ
b_g^ϖ	$\dfrac{F_r^2}{N^2}\dfrac{\partial\hat{\psi}^{\sim\varpi}}{\partial z}\left[\widehat{\mathscr{L}\left(\dfrac{\partial\psi}{\partial z}\right)}\right]^{\sim\varpi}$	Rate of buoyancy conversion on window ϖ
A_g^ϖ	$\dfrac{F_r^2}{2}\dfrac{1}{N^2}\left(\dfrac{\partial\hat{\Psi}^{\sim\varpi}}{\partial z}\right)^2$	QG APE on scale window ϖ
$\mathbf{Q}_{g,A}^\varpi$	$\dfrac{\alpha_\ell}{2}\dfrac{F_r^2}{N^2}\dfrac{\partial\hat{\Psi}^{\sim\varpi}}{\partial z}\left(\widehat{\mathbf{v}_g\dfrac{\partial\Psi}{\partial z}}\right)^{\sim\varpi}$	Flux of QG APE within window ϖ
$\Gamma_{g,A}^\varpi$	$\dfrac{\alpha_\ell}{2}\dfrac{F_r^2}{N^2}\left[\left(\widehat{\mathbf{v}_g\dfrac{\partial\Psi}{\partial z}}\right)^{\sim\varpi}\cdot\nabla_h\dfrac{\partial\hat{\Psi}^{\sim\varpi}}{\partial z} - \dfrac{\partial\hat{\Psi}^{\sim\varpi}}{\partial z}\nabla_h\cdot\left(\widehat{\mathbf{v}_g\dfrac{\partial\Psi}{\partial z}}\right)^{\sim\varpi}\right]$	Canonical transfer of QG APE to window ϖ

In other words, when only $O(\varepsilon)$ is considered, all these terms are negligible. Therefore, the resulting APE equation is, to $O(\varepsilon)$,

$$F_r^2\frac{\partial[A^\varpi]_0}{\partial t} + \alpha_\ell F_r^2\nabla_h \cdot [\mathbf{Q}_{A,h}^\varpi]_0 = \alpha_\ell F_r^2[\Gamma_{A,h}^\varpi]_0 + [b^\varpi]_1. \quad (111)$$

For clarity, this is symbolically written as

$$\frac{\partial}{\partial t}A_g^\varpi + \nabla_h \cdot \mathbf{Q}_{g,A}^\varpi = \Gamma_{g,A}^\varpi + b_g^\varpi, \quad (112)$$

where

$$A_g^\varpi = F_r^2[A^\varpi]_0 = \frac{F_r^2}{2}\frac{1}{N^2}\left(\frac{\partial\hat{\Psi}^{\sim\varpi}}{\partial z}\right)^2,$$

$$\mathbf{Q}_{g,A}^\varpi = \alpha_\ell F_r^2[\mathbf{Q}_{A,h}^\varpi]_0 = \frac{\alpha_\ell}{2}\frac{F_r^2}{N^2}\frac{\partial\hat{\Psi}^{\sim\varpi}}{\partial z}\left(\widehat{\mathbf{v}_g\frac{\partial\Psi}{\partial z}}\right)^{\sim\varpi},$$

$$\Gamma_{g,A}^\varpi = \alpha_\ell F_r^2[\Gamma_{A,h}^\varpi]_0 = \frac{\alpha_\ell}{2}\frac{F_r^2}{N^2}\left[\left(\widehat{\mathbf{v}_g\frac{\partial\Psi}{\partial z}}\right)^{\sim\varpi}\cdot\nabla_h\frac{\partial\hat{\Psi}^{\sim\varpi}}{\partial z} - \frac{\partial\hat{\Psi}^{\sim\varpi}}{\partial z}\nabla_h\cdot\left(\widehat{\mathbf{v}_g\frac{\partial\Psi}{\partial z}}\right)^{\sim\varpi}\right], \quad \text{and}$$

$$b_g^\varpi = [b^\varpi]_1 = \frac{F_r^2}{N^2}\frac{\partial\hat{\psi}^{\sim\varpi}}{\partial z}\left[\widehat{\mathscr{L}\left(\frac{\partial\psi}{\partial z}\right)}\right]^{\sim\varpi}.$$

c. Wrap-up

To summarize, the multiscale energy equations for the inviscid QG equation [(91)] are

$$\frac{\partial}{\partial t}A_g^\varpi + \nabla_h \cdot \mathbf{Q}_{g,A}^\varpi = \Gamma_{g,A}^\varpi + b_g^\varpi \quad \text{and} \quad (113)$$

$$\frac{\partial}{\partial t}K_g^\varpi + \nabla_h \cdot \mathbf{Q}_{g,K}^\varpi = \Gamma_{g,K}^\varpi - \nabla_h \cdot \mathbf{Q}_{g,P,h}^\varpi - \frac{\partial}{\partial z}Q_{g,P,z}^\varpi - b_g^\varpi. \quad (114)$$

The explicit expressions of the energetic terms are tabulated in Table 4. Table 5 gives the canonical transfers in spherical coordinates.

7. Interaction analysis and horizontal treatment

a. Interaction analysis

An energy transfer process toward a certain location in a scale window involves not only the transfer from

TABLE 5. Expansion of the QG canonical transfers in spherical coordinates.

$\Gamma_{K,g}$	$\dfrac{\alpha_\ell}{2a\cos\varphi}\left[\widehat{(u_g^2)}^{\sim\varpi}\dfrac{\partial\hat{u}_g^{\sim\varpi}}{\partial\lambda}-\hat{u}_g^{\sim\varpi}\dfrac{\partial\widehat{(u_g^2)}^{\sim\varpi}}{\partial\lambda}+\widehat{(u_g v_g)}^{\sim\varpi}\dfrac{\partial\hat{v}_g^{\sim\varpi}}{\partial\lambda}-\hat{v}_g^{\sim\varpi}\dfrac{\partial\widehat{(u_g v_g)}^{\sim\varpi}}{\partial\lambda}\right]$
	$+\dfrac{\alpha_\ell}{2a}\left[\widehat{(u_g v_g)}^{\sim\varpi}\dfrac{\partial\hat{u}_g^{\sim\varpi}}{\partial\varphi}-\hat{u}_g^{\sim\varpi}\dfrac{\partial\widehat{(u_g v_g)}^{\sim\varpi}}{\partial\varphi}+\widehat{(v_g^2)}^{\sim\varpi}\dfrac{\partial\hat{v}_g^{\sim\varpi}}{\partial\varphi}-\hat{v}_g^{\sim\varpi}\dfrac{\partial\widehat{(v_g^2)}^{\sim\varpi}}{\partial\varphi}\right]$
	$+\dfrac{3\alpha_\ell}{2a}\tan\varphi[\hat{u}_g^{\sim\varpi}\widehat{(u_g v_g)}^{\sim\varpi}-\hat{v}_g^{\sim\varpi}\widehat{(u_g^2)}^{\sim\varpi}]$
	$+\dfrac{\alpha_\ell}{2a}\hat{v}_g^{\sim\varpi}\tan\varphi[\widehat{(u_g^2)}^{\sim\varpi}+\widehat{(v_g^2)}^{\sim\varpi}]$
$\Gamma_{A,g}$	$\dfrac{\alpha_\ell}{2a\cos\varphi}\dfrac{F_r^2}{N^2}\left\{\widehat{\left(u_g\dfrac{\partial\Psi}{\partial z}\right)}^{\sim\varpi}\dfrac{\partial^2\hat{\Psi}^{\sim\varpi}}{\partial\lambda\partial z}+\widehat{\left(v_g\dfrac{\partial\Psi}{\partial z}\right)}^{\sim\varpi}\dfrac{\partial^2\hat{\Psi}^{\sim\varpi}}{\partial\varphi\partial z}\cos\varphi\right.$
	$\left.-\dfrac{\partial\hat{\Psi}^{\sim\varpi}}{\partial z}\dfrac{\partial\widehat{(u_g\partial\Psi/\partial z)}^{\sim\varpi}}{\partial\lambda}-\dfrac{\partial\hat{\Psi}^{\sim\varpi}}{\partial z}\dfrac{\partial[\widehat{(v_g\partial\Psi/\partial z)}^{\sim\varpi}\cos\varphi]}{\partial\varphi}\right\}$

outside the window but also those from within. This is a fundamental point where it differs from that based on the classical Fourier transform or Reynolds decomposition. Take for an example a transfer[2] Γ_n^1 at location (step) n in window 1. As schematized in Fig. 4, it is the totality of the transfers from window 0, window 2, and those from the other different locations (the sampling space) within the same window. We need to distinguish these subprocesses in order for the window–window interactions to stand out.

As shown above, all the transfers can be written as a linear combination of terms in the form

$$\Gamma_n^\varpi=\hat{\mathscr{R}}_n^{\sim\varpi}\widehat{(pq)}_n^{\sim\varpi}.$$

It therefore suffices to analyze this single term. To make the presentation easier, we here just pick the particular case Γ_n^1. For a detailed treatment, see LR05, section 9. Now what we are considering is the transfer

$$\begin{aligned}\Gamma_n^1&=\hat{\mathscr{R}}_n^{\sim1}\widehat{(pq)}_n^{\sim1}=\hat{\mathscr{R}}_n^{\sim1}\left(\widehat{\sum_{\omega_1=0}^{2}p^{\sim\omega_1}\sum_{\omega_2=0}^{2}q^{\sim\omega_2}}\right)_n^{\sim1}\\&=\hat{\mathscr{R}}_n^{\sim1}[\widehat{(p^{\sim0}q^{\sim0})}_n^{\sim1}+\widehat{(p^{\sim0}q^{\sim1})}_n^{\sim1}+\widehat{(p^{\sim1}q^{\sim0})}_n^{\sim1}]\\&\quad+\hat{\mathscr{R}}_n^{\sim1}[\widehat{(p^{\sim1}q^{\sim2})}_n^{\sim1}+\widehat{(p^{\sim2}q^{\sim1})}_n^{\sim1}+\widehat{(p^{\sim2}q^{\sim2})}_n^{\sim1}]\\&\quad+\hat{\mathscr{R}}_n^{\sim1}[\widehat{(p^{\sim0}q^{\sim2})}_n^{\sim1}+\widehat{(p^{\sim2}q^{\sim0})}_n^{\sim1}]\\&\quad+\hat{\mathscr{R}}_n^{\sim1}\widehat{(p^{\sim1}q^{\sim1})}_n^{\sim1}.\end{aligned}$$

The first two terms represent the energy transfers to scale window 1 from windows 0 and 2, respectively; write them as $\Gamma_n^{0\to1}$ and $\Gamma_n^{2\to1}$. The two-scale windows may also combine to contribute to Γ_n^1, though generally the contribution is negligible; this makes the third term, or $\Gamma_n^{0\oplus2\to1}$ for short. The last term, $\Gamma_n^{1\to1}=\hat{\mathscr{R}}_n^{\sim1}\widehat{(p^{\sim1}q^{\sim1})}_n^{\sim1}$, is the transfer from window 1 itself. The major purpose of interaction analysis is, for scale window 1, to select $\Gamma_n^{0\to1}$ and $\Gamma_n^{2\to1}$ out of Γ_n^1.

For canonical transfers to other scale windows, refer to Table 6.

b. Phase oscillation

The localized multiscale energetics as introduced above may reveal some spurious high-wavenumber oscillation that must be removed. This is a fundamental problem with real-valued localized transforms, which has been carefully examined by Iima and Toh (1995) in the context of shock waves and wavelet analysis. Since
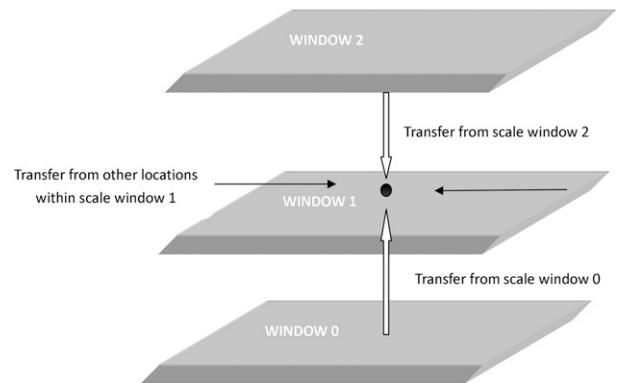

FIG. 4. A schematic of the canonical energy transfer toward scale window 1.

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[2] In this section, the dependence on n is kept in the notations.

TABLE 6. Interaction analysis for Γ^0, Γ^1, and Γ^2.

Γ^2	$\Gamma^{0\to2}$	$\Gamma^{1\to2}$	$\Gamma^{0\oplus1\to2}$	$\Gamma^{2\to2}$
Γ^1	$\Gamma^{0\to1}$	$\Gamma^{2\to1}$	$\Gamma^{0\oplus2\to1}$	$\Gamma^{1\to1}$
Γ^0	$\Gamma^{1\to0}$	$\Gamma^{2\to0}$	$\Gamma^{1\oplus2\to0}$	$\Gamma^{0\to0}$
Remark	Instability related	Instability related	Usually negligible	

this is a technical issue that may prevent one from making the right interpretation, we here give it a brief introduction; for details, refer to LR05.

As others, the MWT transform coefficients contain phase information, and so do the resulting multiscale energies, which are essentially the square of the coefficients. The phase information may not be obvious in the sampling space of the transform coefficients (with elements labeled by n) because of its discrete nature. But the disguised information may appear in the horizontal through a mechanism like Galilean transformation. (In the vertical direction it is negligible because the vertical velocity is generally very weak for geofluid flows.) To illustrate, look at (7), which defines the MWT. The characteristic frequency is $f_c \sim 2^{j_2}$ cycles over the time duration. Let the time step size be Δt, then $f_c \sim 1/\Delta t$. For a flow with speed u_0, the oscillation in time with f_c will result in an oscillation in the horizontal with a wavelength $\sim u_0\Delta t$—that is, a wavenumber $k_c \sim (1/u_0\Delta t)$. Let the mesh size be Δx. For a model to be numerically stable, the CFL condition requires that $\Delta t < \Delta x/u_0$. So the spurious oscillation has a wavenumber $k_c \sim O(1/\Delta x)$.

The phase oscillation is a problem rooted in the nature of localized transforms. In our case, fortunately, it is always around the highest wavenumbers or smallest spatial scales in the spectrum and is, hence, very easy to be removed using, for example, a 2D large-scale window reconstruction (like a horizontal low-pass filtering). This is in contrast to wavelet analysis: the larger the scale for a transform coefficient, the larger the scale for the spurious oscillation (Iima and Toh 1995).

In real applications the spurious oscillation may not show up, just as in the MJO case, which we will demonstrate in the following section. But in some unusual cases this could cause severe errors. We have shown such an example before in LR05 (see the Fig. 2 therein). The analysis is with a simulation of an observed meandering in the Iceland–Faeroe frontal region in August 1993. The mesh grid has a spacing $\Delta x = \Delta y = 2.5$ km, and the time step size is 1800 s. The time series for the multiscale energetics analysis has a sampling interval of $10\Delta t$. So, by the above argument, the phase oscillation, if existing, will have a wavelength less than $10 \times \Delta x = 25$ km. Indeed, as shown in Fig. 2a of LR05, the computed canonical transfer of APE is buried in oscillatory errors, with a wavelength of about eight grid points or

20 km. These errors are efficiently removed through a 2D multiscale window reconstruction with a scale of 25 km; the resulting transfer is shown in their Fig. 2b. (This can also be achieved efficiently using the traditional 2D low-pass filters.)

8. Exemplification with the Madden–Julian oscillation

The above formalism has been validated in previous publications and has seen its success in different real applications. This section is a demonstration of how it may be applied, with the MJO as an example. Note here it is not our intention to perform a comprehensive analysis of the MJO energetics, which will be carefully explored in a forthcoming study.

MJO is a coupled convection–circulation phenomenon, manifesting itself as a localized structure of enhanced and suppressed precipitation propagating in the zonal direction at a speed of 4–8 m s^{-1} (cf. Fig. 5). It is the largest element of intraseasonal variability in the tropical atmosphere (Madden and Julian 1971). Though extending through the whole tropics, the anomalous rainfall occurs mainly over the Indian Ocean and western Pacific Ocean. The oscillation has a broadband spectrum between the 30- and 60-day periods. It is usually strong in winter and spring and weak in summer. By observation, it originates over the western Indian Ocean, strengthens as it enters the western Pacific, and dies out east of the date line. According to Wheeler and Hendon (2004), a complete MJO cycle comprises eight phases, each corresponding to the position of the center of the anomalous rainfall, from western Indian Ocean to eastern Pacific Ocean. As an intraseasonal phenomenon, MJO bridges the large-scale and small-scale motions in the atmospheric spectrum, making an important component of the atmospheric circulation. Various studies have established its connections to tropical cyclogenesis, El Niño–Southern Oscillation, and South Asia monsoon, to name a few [see Madden and Julian (2005) and the references therein].

With large-scale atmospheric circulation and tropical deep convection intricately coupled, MJO provides an excellent example for the study of multiscale interaction. Analytical investigations of the interaction has been made available in the systematic work of Majda et al. (e.g., Majda and Biello 2004; Majda and
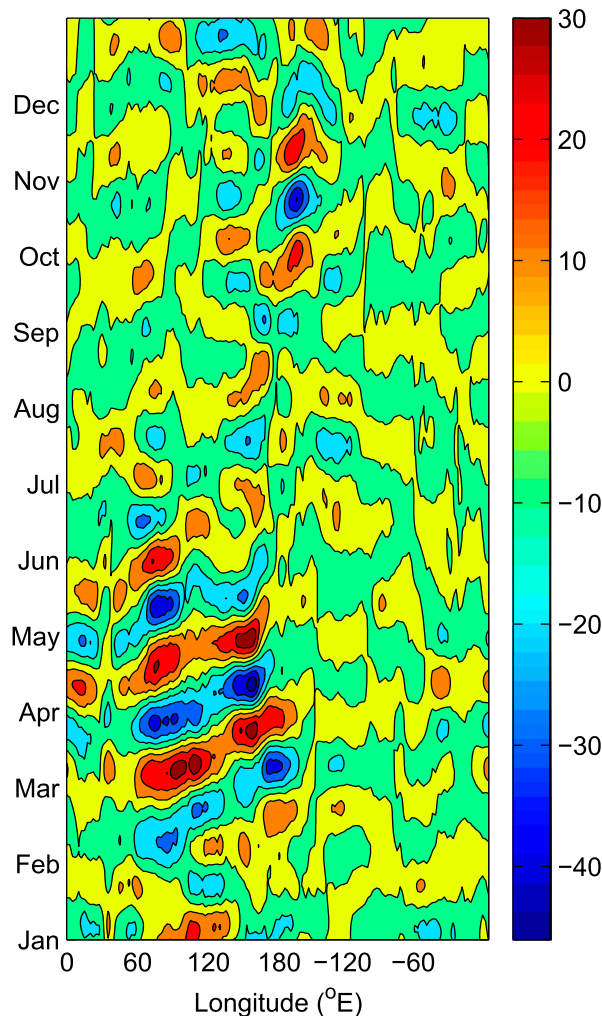
FIG. 5. The 32–64-day scale-window reconstruction for the 1997 OLR anomaly (W m^{-2}) in the tropical area (averaged over 10°S–10°N).

Yang 2016). Notice the localized and progressive pattern: it makes MJO an ideal test bed for our formalism of multiscale energetics. We are therefore using it for our purpose of demonstration.

The data we are using include those from the European Centre for Medium-Range Weather Forecasts (ECMWF) interim reanalysis (ERA-Interim; http://www.ecmwf.int/en/research/climate-reanalysis/era-interim) daily products (wind, temperature, and geopotential height) and the series of the real-time multivariate MJO (RMM) (Wheeler and Hendon 2004). They have a spatial resolution of 2.5° × 2.5° and span from 1988 through 2010. The vertical temperature profile, $\overline{T} = \overline{T}(p)$, which is needed in the application, is obtained by taking the time mean of T, followed by an averaging over all the p planes.

To begin, we need to demarcate the scale windows. The problem forms a natural three-window decomposition:

large-scale variabilities, MJO, and synoptic processes. We choose an MJO window of 32–64 days, since in the analysis a power of 2 for a window bound is required.

We choose a strong MJO event on 16 December 1996 for our exemplification purpose. The RMM index is 2.05, corresponding to phase 5 (where the convection center is over the Maritime Continent). Using the above parameters, a straightforward application to the outgoing longwave radiation (OLR) in the tropical region (averaged between 10°S and 10°N) immediately yields an MJO window reconstruction (Fig. 5). From it, the eastward propagation and its seasonal variation are clearly seen. Likewise, velocity and temperature can be reconstructed. Particularly, $u^{\sim 1}$, $\omega^{\sim 1}$, and $T^{\sim 1}$ have on the zonal cross section an up-westward-tilting pattern, as identified earlier on (e.g., Moncrieff 2004); see Fig. 6.

Shown in Fig. 7 are the vertical distributions of the canonical transfers to the MJO window averaged over the tropical region (10°S–10°N) between 0° and 180°. From the kinetic transfers, $\Gamma_K^{0 \to 1}$ is on the whole positive, while $\Gamma_K^{2 \to 1}$ is negative. That is to say, Γ_K is downscale. In contrast, its potential energy counterpart tends to be more irregularly distributed, and, besides, is one order smaller. Though this is just for one particular day only, the long time mean also has the trend. This is in opposite to that for the midlatitude paradigm, where the canonical APE transfer is downscale while the canonical KE transfer is upscale (Saltzman 1970). From the figure the transfer center is located in the upper troposphere around 200 hPa, in agreement with the previous studies (e.g., Hsu et al. 2011).

To examine the horizontal distributions of instability centers, in Fig. 8 we draw the maps of the canonical transfers at 200 hPa. We see that they are mainly distributed between 100° and 140°E—that is, the Maritime Continent. This is, of course, in agreement with the phase where MJO lies at that time.

We emphasize again that it is not our intention to study the MJO dynamics here. We just pick for the purpose of demonstration this example at a given instance. It is seen that, through a straightforward application, one immediately obtains a bunch of maps of the multiscale energetics that reflect the underlying internal dynamics, and these energetics agree well with the previous studies. A detailed study of MJO the intraseasonal mode requires a statistical examination of the resulting energetics; we will see that later in Lu et al. (2016, unpublished manuscript).

9. Conclusions and discussion

Multiscale energetics diagnostics are important in that they provide an approach to the fundamental problems of atmospheres and oceans such as mean flow–disturbances interaction, instability, and disturbance growth, as
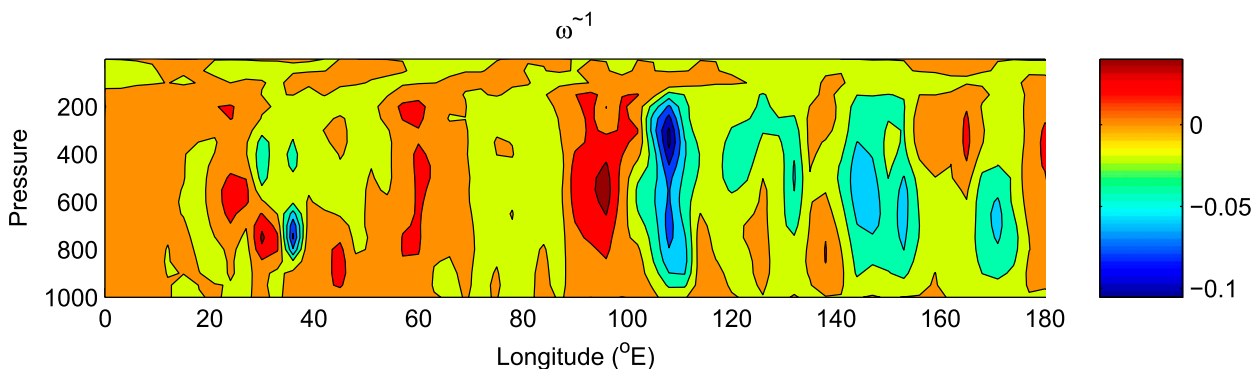
FIG. 6. The equatorial $\omega = dp/dt$ on 16 Dec 1996 reconstructed on the 32–64-day scale window. Note the up-westward-tilting pattern east of the Maritime Continent.

identified in the national report of Lindzen and Farrell (1987). Their importance is also seen in the potential role that they may play in the major engineering problems such as eddy transport parameterization (e.g., Gent and McWilliams 1990; Greatbatch 1998; Visbeck et al. 1997; Marshall and Adcroft 2010) and turbulence and feedback closure (e.g., Jin 2010). Based on the new analysis machinery, namely, multiscale window transform (MWT), which is capable of orthogonally decomposing a function space into a direct sum of several subspaces while retaining the local information in the resulting transform

coefficients, we have given a comprehensive derivation of the multiscale energetics for the atmosphere, with respect to both the primitive equation model and quasigeostrophic model. By taking advantage of the nice properties of the MWT, an "atomic" reconstruction of the fluxes on the multiscale windows allows for a unique separation of the interscale transfer from the nonlinearly intertwined energetics. The resulting transfer bears a Lie bracket form, reminiscent of the Poisson bracket in the Hamiltonian dynamics; for this reason, we call it canonical transfer. A canonical transfer sums to zero over scale windows, indicating that it is a mere
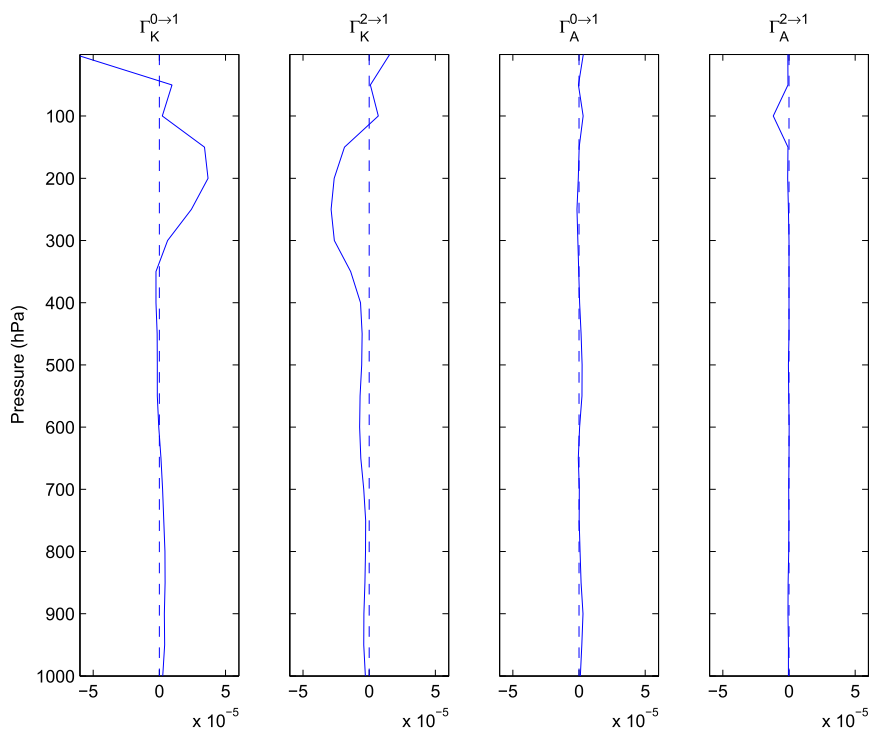

FIG. 7. Vertical distributions of $\Gamma_K^{0\to1}$, and $\Gamma_K^{2\to1}$, $\Gamma_A^{0\to1}$, and $\Gamma_A^{2\to1}$ (all in $m^2\,s^{-3}$) averaged between 10°S and 10°N and 0° and 180°.
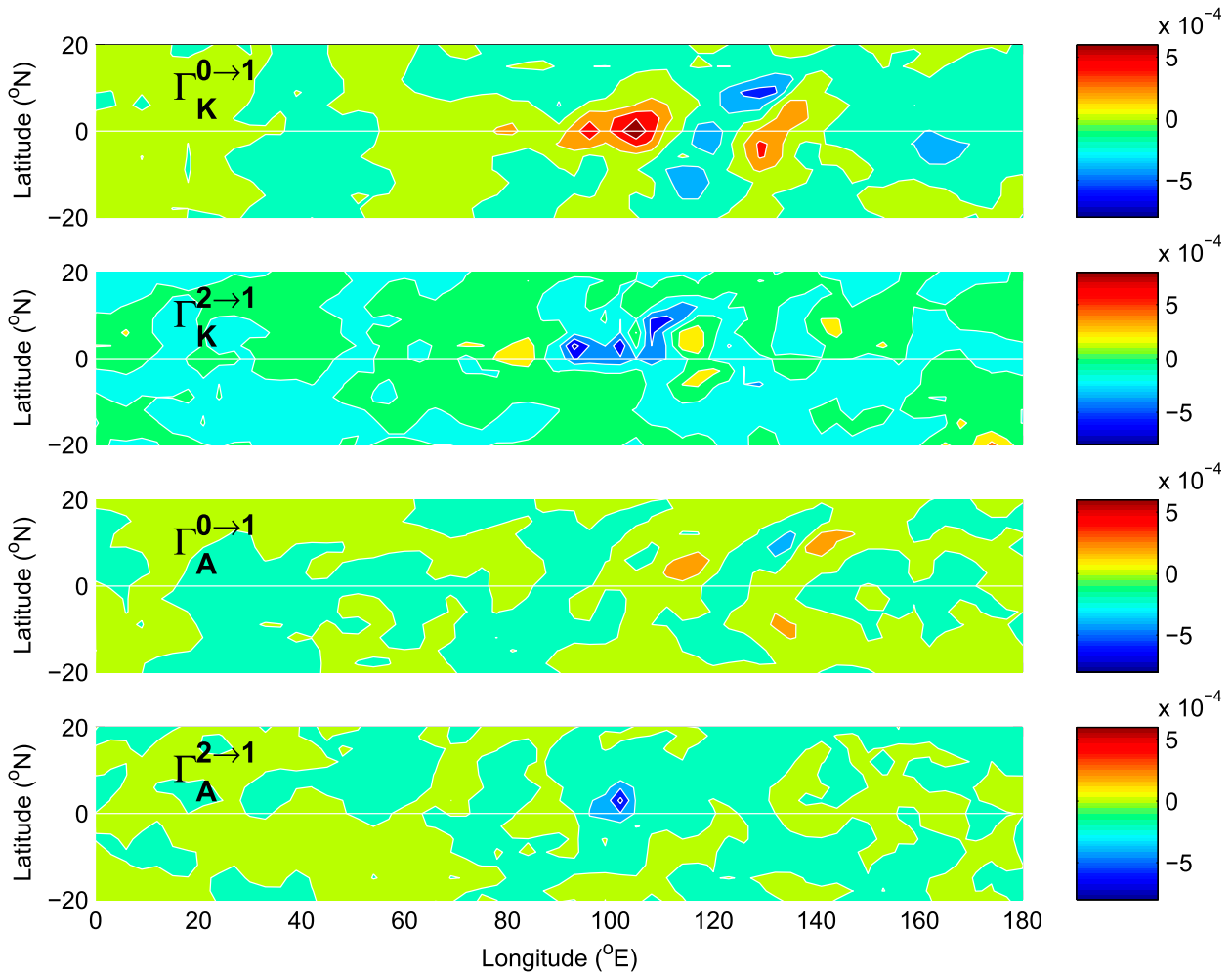
FIG. 8. Horizontal distributions of $\Gamma_K^{0\to1}$, $\Gamma_K^{2\to1}$, $\Gamma_A^{0\to1}$, and $\Gamma_A^{2\to1}$ (all in $m^2\,s^{-3}$) at 100 hPa.

redistribution of energy among the scale windows, without generating or destroying energy as a whole.

The multiscale atmospheric kinetic energy (KE) and available potential energy (APE) equations are thence derived. By classification, a multiscale energetic cycle comprises the following processes: KE transport, APE transport, pressure work, buoyancy conversion, work done by external forcing and diabatic and frictional processes in the respective scale windows, and the interscale canonical transfers of KE and APE, which have been shown to correspond to the barotropic and baroclinic instabilities (Liang and Robinson 2007). Note that a buoyancy conversion takes place in an individual window only, bridging the two types of energy—namely, KE and APE. It does not involve the process among different scale windows and, hence, basically is not related to instabilities, although traditionally it has been used to diagnose baroclinic instabilities. A brief application of the formalism is exemplified with the Madden–Julian oscillation.

Also derived are the multiscale KE and APE equations for quasigeostrophic flows and, for completeness, those for oceanic circulations. It should be cautioned that, since what we talk about are four-dimensional energy distribution and evolution, the term "energy" in this study is, in a strict sense, "energy density." The abuse of terminology will not cause confusion as it is clear in the context.

It should be mentioned that the definition of APE is still an active arena of research; a recent review can be found in Tailleux (2013). In the present formalism, APE is defined as in Lorenz (1955), which takes a quadratic form. However, it has been argued that it is generally not quadratic, if the 1D reference hydrostatic thermodynamic profile is achieved by adiabatic rearrangement of the existing 3D state (e.g., Holliday and McIntyre 1981; Winters et al. 1995; Winters and Barkan 2013). This raises an issue about how to handle APE in nonquadratic form in the multiscale formalism. Recall that, in this study, central at the

multiscale energy representation is the Parseval relation, while the relation works only for quadratic properties. For a nonquadratic APE, the problem may need to be considered from a more fundamental point of view. We will leave that to future discussions.

Notice that presented in this study is the energetics based on a three-scale window decomposition. It is straightforward to extend the formalism to four, five, or more scale windows; the resulting energy equations are the same in form. One may equally reduce the number of windows to two.

We remark that there is a well-known apparatus in achieving a two-scale decomposition in atmospheric research—that is, decomposition through taking the transformed Eulerian mean (Andrews and McIntyre 1976; McIntyre 1980; also see Plumb and Ferrari 2005; Bühler 2009). Formalisms of two-scale energetics have been established with the theory (e.g., Plumb 1983), but how these formalisms may be related to this study has yet to be carefully examined.

In LR05, there is also a brief touch on multiscale enstrophy analysis, which, together with its multiscale energetics counterpart, makes "localized multiscale energy and vorticity analysis," or MS-EVA as called therein. Since the multiscale enstrophy equation is closely related to an important concept in dynamic meteorology—namely, the Eliassen–Palm flux (Eliassen and Palm 1961; Bühler 2009; Vallis 2006), which has been extensively employed in wave activity diagnosis and certainly deserves a detailed study for its own sake (e.g., Marshall 1984; Plumb 1986; Rhines and Holland 1979; Nakamura and Solomon 2010; Takaya 2001)—we will defer it to another investigation in the near future.

APPENDIX A

Glossary of Notation

∇	3D gradient operator $[=\mathbf{e}_\lambda(\partial/\partial\lambda) + \mathbf{e}_\varphi(\partial/\partial\varphi) + \mathbf{e}_p(\partial/\partial p)]$
∇_h	Horizontal gradient operator (horizontal component of ∇)
$\mathbf{v}$	Velocity $[=(u, v, \omega), \omega = dp/dt$ (atmosphere); $=(u, v, w)$ (ocean)]
$\mathbf{v}_h$	Horizontal velocity $[=(u, v)]$
ϕ	Scaling function
$\hat{\pi}^{\sim\varpi}$	MWT of some property π at step n on window ϖ; dependence on n is suppressed when no confusion arises
$\pi^{\sim\varpi}$	Window ϖ-filtered π (multiscale window reconstruction of π on window ϖ)
π_n^ϖ	Notation of some property at step n on window ϖ; n is suppressed when no confusion arises
$\overline{T} = \overline{T}(p)$	Mean temperature profile (averaged over the p plane and time)
T	Departure from $\overline{T}$
α	Specific volume
Φ	Geopotential function
Z	Geopotential height $(=\Phi/g)$
R	Specific gas constant $(\mathrm{J\,kg^{-1}\,K^{-1}})$
c_p	Specific heat capacity of air for isobaric processes $(=1.005 \times 10^3\,\mathrm{J\,kg^{-1}\,K^{-1}})$
c_v	Specific heat capacity of air for isochoric processes
f	Coriolis parameter
β	Meridional gradient of f
L	Lapse rate $(=-\partial\overline{T}/\partial z)$
L_d	Lapse rate of dry air $(=g/c_p)$
a	Radius of Earth
(λ, φ, r)	Spherical coordinates
p	Pressure coordinate
(x, y, z)	Zonal arc length, meridional arc length, and height measured from Earth's surface $(z = r - a)$; $dx = a\cos\varphi d\lambda$, $dy = ad\varphi$
$(\tilde{x}, \tilde{y}, \tilde{z})$	Cartesian coordinates
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	Unit vectors for the Cartesian coordinate system
$\mathbf{e}_\lambda, \mathbf{e}_\varphi, \mathbf{e}_z$	Unit vectors for spherical coordinate system
$\mathbf{e}_\lambda, \mathbf{e}_\varphi, \mathbf{e}_p$	Unit vectors for the isobaric spherical coordinate system
g	Acceleration due to gravity
$(h_\lambda, h_\varphi, h_z)$	Lamé's coefficients
$\overline{\rho} = \overline{\rho}(z)$	Stationary density profile (ocean)
ρ	Density perturbation with $\overline{\rho}$ removed (ocean)
ρ_0	Reference density $(=1025\,\mathrm{kg\,m^{-3}}$ here) (ocean)
N	Buoyancy frequency (ocean) $[=N(z) = \sqrt{-(g/\rho_0)(\partial\overline{\rho}/\partial z)}\,]$
P	Dynamic pressure; that is, pressure with $\overline{P}(z) = \overline{P}_0 - \int_0^z \overline{\rho}g\,dz$ removed (ocean)
c	Premultiplier for available potential energy $\{=g/[\overline{T}(g/c_p - L)]$ (atmosphere); $=g^2/\rho_0^2 N^2$ (ocean)$\}$
$\mathbf{Q}$	Flux
Γ	Canonical transfer
A	Available potential energy
K	Kinetic energy
b	Buoyancy conversion rate
F_h	Friction/external forcing in horizontal direction

F_z	Friction/external forcing in vertical direction
F_p	Friction/external forcing in p direction
ψ	Streamfunction
$\mathbf{v}_g$	Geostrophic velocity ($=\mathbf{k} \times \nabla_h \psi$)
F_r	Rotational internal Froude number
ε	Rossby number ($=1/f_0 t_0$)
α_ℓ	Measure of importance of advection to local change ($=U_0 t_0/L_0$)
$\mathscr{L}$	Substantial differential operator with respect to the geostrophic flow $[=\partial/\partial t + \mathbf{v}_g \cdot \nabla_h = (\partial/\partial t) + J(\psi, \cdot)]$

APPENDIX B

Expansion of $\nabla \cdot (\mathbf{vv})$ in Spherical Coordinates

To compute the canonical transfer [(46)], we are required to evaluate explicitly $\nabla \cdot (\mathbf{vv}_h)$ in the spherical coordinate system (λ, φ, r). This material, which is often missing or appears incomplete in textbooks, is hopefully of some use to researchers.

The coordinates (λ, φ, r) are connected with the Cartesian coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ as follows:

$$\tilde{x} = r \cos\varphi \cos\lambda, \tag{B1}$$

$$\tilde{y} = r \cos\varphi \sin\lambda, \quad \text{and} \tag{B2}$$

$$\tilde{z} = r \sin\varphi. \tag{B3}$$

Here the tilde is employed to avoid confusing with z, which will be reserved for height measured from Earth's surface: $z = r - a$, with a being the radius of Earth. Besides, in meteorology, dx and dy are usually reserved for $a\cos\varphi \, d\lambda$ and $a \, d\varphi$, respectively. From the position
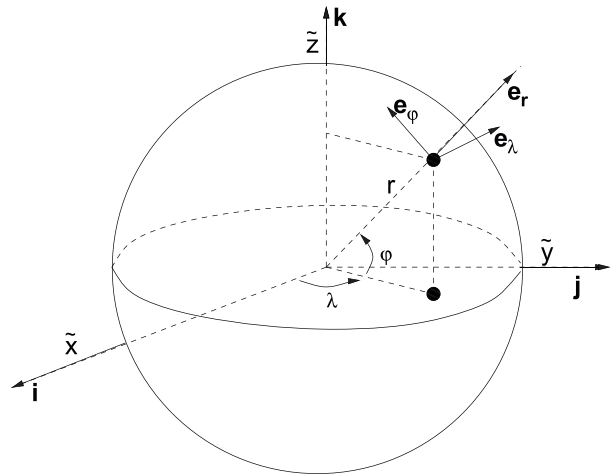

FIG. B1. Spherical coordinate frame.

vector $\mathbf{x} = \tilde{x}\mathbf{i} + \tilde{y}\mathbf{j} + \tilde{z}\mathbf{k}$, it is easy to find the Lamé's coefficients as follows (cf. Fig. B1):

$$h_\lambda = \left|\frac{\partial \mathbf{x}}{\partial \lambda}\right| = r\cos\varphi, \tag{B4}$$

$$h_\varphi = \left|\frac{\partial \mathbf{x}}{\partial \varphi}\right| = r, \quad \text{and} \tag{B5}$$

$$h_z = \left|\frac{\partial \mathbf{x}}{\partial z}\right| = 1. \tag{B6}$$

With the shallow-water approximation, $r \approx a =$ constant. So

$$\nabla \cdot (\mathbf{vv}) = \nabla \cdot [\mathbf{v}(u\mathbf{e}_\lambda + v\mathbf{e}_\varphi + w\mathbf{e}_z)]$$
$$= \frac{1}{a\cos\varphi}\frac{\partial[u(u\mathbf{e}_\lambda + v\mathbf{e}_\varphi + w\mathbf{e}_z)]}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial[v(u\mathbf{e}_\lambda + v\mathbf{e}_\varphi + w\mathbf{e}_z)\cos\varphi]}{\partial\varphi} + \frac{\partial[w(u\mathbf{e}_\lambda + v\mathbf{e}_\varphi + w\mathbf{e}_z)]}{\partial z}. \tag{B7}$$

And, particularly,

$$\nabla \cdot (\mathbf{vv}_h) = \nabla \cdot [\mathbf{v}(u\mathbf{e}_\lambda + v\mathbf{e}_\varphi)]$$
$$= \frac{1}{a\cos\varphi}\frac{\partial[u(u\mathbf{e}_\lambda + v\mathbf{e}_\varphi)]}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial[v(u\mathbf{e}_\lambda + v\mathbf{e}_\varphi)\cos\varphi]}{\partial\varphi} + \frac{\partial[w(u\mathbf{e}_\lambda + v\mathbf{e}_\varphi)]}{\partial z}. \tag{B8}$$

We need to evaluate $\partial\mathbf{e}_\lambda/\partial\lambda$, $\partial\mathbf{e}_\lambda/\partial\varphi$, etc.

There are several ways to achieve the evaluation. One way is by directly taking the limit $\partial\mathbf{e}_\lambda/\partial\lambda = \lim_{\Delta\lambda\to 0}(\Delta\mathbf{e}_\lambda/\Delta\lambda)$. Another way is to first connect $(\mathbf{e}_\lambda, \mathbf{e}_\varphi, \mathbf{e}_z)$ with $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, then take the derivatives. One may take advantage of the properties such as

$$\mathbf{e}_\lambda \cdot \mathbf{e}_\varphi = 1 \Rightarrow \partial\mathbf{e}_\lambda \cdot \mathbf{e}_\lambda = 0 \Rightarrow \partial\mathbf{e}_\lambda \perp \mathbf{e}_\lambda.$$

From Fig. B1, it is easy to find that

$$\mathbf{e}_\lambda = -\sin\lambda\mathbf{i} + \cos\lambda\mathbf{j}, \tag{B9}$$

$$\mathbf{e}_\varphi = -\sin\varphi\cos\lambda\mathbf{i} - \sin\varphi\sin\lambda\mathbf{j} + \cos\varphi\mathbf{k}, \quad \text{and} \tag{B10}$$

$$\mathbf{e}_z = \cos\varphi\cos\lambda\mathbf{i} + \cos\varphi\sin\lambda\mathbf{j} + \sin\varphi\mathbf{k}. \tag{B11}$$

Inverting, we get

$$\mathbf{i} = -\cos\lambda\,\sin\varphi\mathbf{e}_\varphi + \cos\lambda\,\cos\varphi\mathbf{e}_z - \sin\lambda\mathbf{e}_\lambda, \tag{B12}$$

$$\mathbf{j} = -\sin\lambda\,\sin\varphi\mathbf{e}_\varphi + \sin\lambda\,\cos\varphi\mathbf{e}_z + \cos\lambda\mathbf{e}_\lambda, \quad\text{and} \tag{B13}$$

$$\mathbf{k} = \cos\varphi\mathbf{e}_\varphi + \sin\varphi\mathbf{e}_z. \tag{B14}$$

So

$$\frac{\partial\mathbf{e}_\lambda}{\partial\lambda} = -\cos\lambda\mathbf{i} - \sin\lambda\mathbf{j} = \sin\varphi\mathbf{e}_\varphi - \cos\varphi\mathbf{e}_z, \tag{B15}$$

$$\frac{\partial\mathbf{e}_\lambda}{\partial\varphi} = 0, \tag{B16}$$

$$\frac{\partial\mathbf{e}_\lambda}{\partial z} = 0, \tag{B17}$$

$$\frac{\partial\mathbf{e}_\varphi}{\partial\lambda} = \sin\varphi\,\sin\lambda\mathbf{i} - \sin\varphi\,\cos\lambda\mathbf{j} = -\sin\varphi\mathbf{e}_\lambda, \tag{B18}$$

$$\frac{\partial\mathbf{e}_\varphi}{\partial\varphi} = -\cos\varphi\,\cos\lambda\mathbf{i} - \cos\varphi\,\sin\lambda\mathbf{j} - \sin\varphi\mathbf{k} = -\mathbf{e}_z, \tag{B19}$$

$$\frac{\partial\mathbf{e}_\varphi}{\partial z} = 0, \tag{B20}$$

$$\frac{\partial\mathbf{e}_z}{\partial\lambda} = -\cos\varphi\,\sin\lambda\mathbf{i} + \cos\varphi\,\cos\lambda\mathbf{j} = \cos\varphi\mathbf{e}_\lambda, \tag{B21}$$

$$\frac{\partial\mathbf{e}_z}{\partial\varphi} = -\sin\varphi\,\cos\lambda\mathbf{i} - \sin\varphi\,\sin\lambda\mathbf{j} + \cos\varphi\mathbf{k} = \mathbf{e}_\varphi, \tag{B22}$$

and

$$\frac{\partial\mathbf{e}_z}{\partial z} = 0. \tag{B23}$$

Also one may obtain

$$\frac{d\mathbf{e}_\lambda}{dt} = \frac{u}{a\cos\varphi}(\mathbf{e}_\varphi\,\sin\varphi - \mathbf{e}_z\,\cos\varphi), \tag{B24}$$

$$\frac{d\mathbf{e}_\varphi}{dt} = \frac{u\tan\varphi}{a}\mathbf{e}_\lambda - \frac{v}{a}\mathbf{e}_z, \quad\text{and} \tag{B25}$$

$$\frac{d\mathbf{e}_z}{dt} = \frac{u}{a}\mathbf{e}_\lambda + \frac{v}{a}\mathbf{e}_z. \tag{B26}$$

With the above results, (B7) now can be expanded as

$$\nabla\cdot(\mathbf{vv}) = \frac{1}{a\cos\varphi}\left(\frac{\partial u^2}{\partial\lambda}\mathbf{e}_\lambda + \frac{\partial uv}{\partial\lambda}\mathbf{e}_\varphi + \frac{\partial uw}{\partial\lambda}\mathbf{e}_z + u^2\frac{\partial\mathbf{e}_\lambda}{\partial\lambda} + uv\frac{\partial\mathbf{e}_\varphi}{\partial\lambda} + uw\frac{\partial\mathbf{e}_z}{\partial\lambda}\right)$$
$$+ \frac{1}{a\cos\varphi}\left[\frac{\partial(vu\cos\varphi)}{\partial\varphi}\mathbf{e}_\lambda + \frac{\partial(v^2\cos\varphi)}{\partial\varphi}\mathbf{e}_\varphi + \frac{\partial(vw\cos\varphi)}{\partial\varphi}\mathbf{e}_z + vu\cos\varphi\frac{\partial\mathbf{e}_\lambda}{\partial\varphi} + v^2\cos\varphi\frac{\partial\mathbf{e}_\varphi}{\partial\varphi} + vw\cos\varphi\frac{\partial\mathbf{e}_z}{\partial\varphi}\right]$$
$$+ \left(\frac{\partial wu}{\partial z}\mathbf{e}_\lambda + \frac{\partial wv}{\partial z}\mathbf{e}_\varphi + \frac{\partial w^2}{\partial z}\mathbf{e}_z + wu\frac{\partial\mathbf{e}_\lambda}{\partial z} + wv\frac{\partial\mathbf{e}_\varphi}{\partial z} + w^2\frac{\partial\mathbf{e}_z}{\partial z}\right). \tag{B27}$$

Or,

$$\nabla\cdot(\mathbf{vv}) = \left\{\frac{1}{a\cos\varphi}\left[\frac{\partial u^2}{\partial\lambda} - uv\sin\varphi + uw\cos\varphi + \frac{\partial(vu\cos\varphi)}{\partial\varphi}\right] + \frac{\partial wu}{\partial z}\right\}\mathbf{e}_\lambda$$
$$+ \left\{\frac{1}{a\cos\varphi}\left[\frac{\partial uv}{\partial\lambda} + u^2\sin\varphi + \frac{\partial(v^2\cos\varphi)}{\partial\varphi} + vw\cos\varphi\right] + \frac{\partial wv}{\partial z}\right\}\mathbf{e}_\varphi$$
$$+ \left\{\frac{1}{a\cos\varphi}\left[\frac{\partial uw}{\partial\lambda} - u^2\cos\varphi + \frac{\partial(vw\cos\varphi)}{\partial\varphi} - v^2\cos\varphi\right] + \frac{\partial w^2}{\partial z}\right\}\mathbf{e}_z. \tag{B28}$$

One may check that, with the aid of the incompressibility assumption

$$\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial v\cos\varphi}{\partial\varphi} + \frac{\partial w}{\partial z} = 0,$$

the above equation is equivalent to

$$\left(\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{v}{a}\frac{\partial u}{\partial\varphi} + w\frac{\partial u}{\partial z} - \frac{uv}{a}\tan\varphi + \frac{uw}{a}\right)\mathbf{e}_\lambda$$
$$+ \left(\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{u^2}{a}\tan\varphi + \frac{v}{a}\frac{\partial v}{\partial\varphi} + \frac{vw}{a} + w\frac{\partial v}{\partial z}\right)\mathbf{e}_\varphi$$
$$+ \left(\frac{u}{a\cos\varphi}\frac{\partial w}{\partial\lambda} + \frac{v}{a}\frac{\partial w}{\partial\varphi} + w\frac{\partial w}{\partial z} - \frac{u^2 + v^2}{a}\right)\mathbf{e}_z,$$

which is precisely the advection part in the non-approximated momentum equations in spherical coordinates. Equation (B28) is thence verified.

As a particular case,

$$\nabla \cdot (\mathbf{v}\mathbf{v}_h) = \frac{1}{a\cos\varphi}\left(\frac{\partial u^2}{\partial \lambda}\mathbf{e}_\lambda + \frac{\partial uv}{\partial \lambda}\mathbf{e}_\varphi + u^2\frac{\partial \mathbf{e}_\lambda}{\partial \lambda} + uv\frac{\partial \mathbf{e}_\varphi}{\partial \lambda}\right)$$
$$+ \frac{1}{a\cos\varphi}\left[\frac{\partial(vu\cos\varphi)}{\partial\varphi}\mathbf{e}_\lambda + \frac{\partial(v^2\cos\varphi)}{\partial\varphi}\mathbf{e}_\varphi\right.$$
$$\left. + vu\cos\varphi\frac{\partial\mathbf{e}_\lambda}{\partial\varphi} + v^2\cos\varphi\frac{\partial\mathbf{e}_\varphi}{\partial\varphi}\right]$$
$$+ \left(\frac{\partial wu}{\partial z}\mathbf{e}_\lambda + \frac{\partial wv}{\partial z}\mathbf{e}_\varphi + wu\frac{\partial\mathbf{e}_\lambda}{\partial z} + wv\frac{\partial\mathbf{e}_\varphi}{\partial z}\right).$$
(B29)

Or,

$$\nabla \cdot (\mathbf{v}\mathbf{v}_h) = \left\{\frac{1}{a\cos\varphi}\left[\frac{\partial u^2}{\partial\lambda} - uv\sin\varphi + \frac{\partial(vu\cos\varphi)}{\partial\varphi}\right] + \frac{\partial wu}{\partial z}\right\}\mathbf{e}_\lambda$$
$$+ \left\{\frac{1}{a\cos\varphi}\left[\frac{\partial uv}{\partial\lambda} + u^2\sin\varphi + \frac{\partial(v^2\cos\varphi)}{\partial\varphi}\right] + \frac{\partial wv}{\partial z}\right\}\mathbf{e}_\varphi$$
$$+ \left[\frac{1}{a\cos\varphi}(-u^2\cos\varphi - v^2\cos\varphi)\right]\mathbf{e}_z.$$
(B30)

Correspondingly with the incompressibility assumption, this is

$$\left(\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{v}{a}\frac{\partial u}{\partial\varphi} + w\frac{\partial u}{\partial z} - \frac{uv}{a}\tan\varphi\right)\mathbf{e}_\lambda$$
$$+ \left(\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{u^2}{a}\tan\varphi + \frac{v}{a}\frac{\partial v}{\partial\varphi} + w\frac{\partial v}{\partial z}\right)\mathbf{e}_\varphi$$
$$+ \left(-\frac{u^2 + v^2}{a}\right)\mathbf{e}_z.$$

APPENDIX C

Some Quasigeostrophic Results Used in the Text

Using the scaling in section 6, it is easy to have the scaled inviscid governing equations [(71)–(74)] as follows (now all the variables in this appendix are understood as nondimensional):

$$\varepsilon\frac{\partial\mathbf{v}_h}{\partial t} + \varepsilon\alpha_\ell\left(\mathbf{v}_h\cdot\nabla_h\mathbf{v}_h + w\frac{\partial\mathbf{v}_h}{\partial z}\right) + f\mathbf{k}\times\mathbf{v}_h = -\nabla_h P,$$
(C1)

$$\rho = -\frac{\partial P}{\partial z},$$
(C2)

$$\nabla_h\cdot\mathbf{v}_h + \frac{\partial w}{\partial z} = 0, \quad \text{and} \quad$$
(C3)

$$F_r^2\varepsilon\frac{\partial\rho}{\partial t} + F_r^2\varepsilon\alpha_\ell\left(\mathbf{v}_h\cdot\nabla_h\rho + w\frac{\partial\rho}{\partial z}\right) = N^2 w, \quad$$
(C4)

where $f = 1 + \varepsilon\beta y$.

Expanding P, w, $\mathbf{v}_h$, and ρ in the power of ε, as that in (96)–(99), it is easy to show that

$$[\mathbf{v}_h]_0 = \mathbf{k}\times\nabla[P]_0,$$
(C5)

$$[w]_0 = 0, \quad \text{and} \quad$$
(C6)

$$[\rho]_0 = -\frac{\partial[P]_0}{\partial z}$$
(C7)

and

$$[w]_1 = -\frac{F_r^2}{N^2}\mathscr{L}\left(\frac{\partial[P]_0}{\partial z}\right) \quad \text{and} \quad$$
(C8)

$$[\mathbf{v}_h]_1 = \mathbf{k}\times\mathscr{L}([\mathbf{v}_h]_0) - \beta y[\mathbf{v}_h]_0 + \mathbf{k}\times\nabla_h[P]_1, \quad$$
(C9)

where $\mathscr{L}$ is the substantial differential operator along the geostrophic flow $[\mathbf{v}_h]_0$: $\mathscr{L} = \partial/\partial t + [\mathbf{v}_h]_0\cdot\nabla_h$. Equations (C5)–(C9) are to be used in the text in section 6.

As is conventional, let $[P]_0 \equiv \psi$. Following the derivations in standard textbooks (e.g., McWilliams 2006), we have

$$\frac{\partial}{\partial t}\left[\nabla_h^2\psi + \frac{\partial}{\partial z}\left(\frac{F_r^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right]$$
$$+ \alpha_\ell J\left\{\psi, \left[\nabla_h^2\psi + \frac{\partial}{\partial z}\left(\frac{F_r^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right]\right\} + \beta\frac{\partial\psi}{\partial x} = 0, \quad$$
(C10)

where J is the Jacobian operator. This is the very quasigeostrophic equation for which we derive the multiscale energetics.

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